Part I. (Do 3 problems)

1. Solve
\[
\begin{cases}
\frac{\partial u}{\partial x} + x \frac{\partial u}{\partial y} = u \\
u(1,y) = h(y)
\end{cases}
\]
where \( h \in C^1(\mathbb{R}) \).

2. Let \( k \in \mathbb{R} \) and let
\[\Gamma(x) = \frac{e^{ik|x|}}{|x|} \quad x \in \mathbb{R}^3, x \neq 0.\]
Prove that \( \Gamma \) satisfies the Helmholtz equation \( \Delta \Gamma + k^2 \Gamma = 0 \) for \( x \neq 0 \).

3. Let \( f \in L^1(\mathbb{R}^n) \) and its Fourier transform \( \hat{f}(x) = \int_{\mathbb{R}^n} f(y) e^{-2\pi i x y} \, dy \). If \( g(x) = |x| \, f(x) \) belongs to \( L^1(\mathbb{R}^n) \), then prove that \( \hat{f} \) satisfies the Lipschitz estimate
\[|\hat{f}(x) - \hat{f}(y)| \leq 2\pi |g|_1 |x-y| \quad \forall x, y \in \mathbb{R}^n.\]

4. Let \( F, G : \mathbb{R} \to \mathbb{R} \) be continuous and let \( w(\xi, \eta) = F(\xi) + G(\eta) \). Prove that \( w \) is a generalized solution to the equation \( w_{\xi \eta} = 0 \), that is,
\[\int_{\mathbb{R}^2} w(\xi, \eta) \phi_{\xi \eta}(\xi, \eta) \, d\xi d\eta = 0 \quad \forall \phi \in C_0^2(\mathbb{R}^2).\]
Conclude that \( u(x,t) = F(x+ct) + G(x-ct) \) is a generalized solution to the wave equation \( \square u = u_{tt} - \Delta u = 0 \), that is,
\[\int_{\mathbb{R}^2} u(x,t) \, \square \phi(x,t) \, dx dt = 0 \quad \forall \phi \in C_0^2(\mathbb{R}^2).\]

Part II. (Do 2 problems)

1. Let \( u(x,t) \) be a \( C^2 \) bounded solution of
\[u_t(x,t) - u_{xx}(x,t) = 0, \quad x \in \mathbb{R}, \ t > 0, \ u(x,0) = f(x)\]
where \( f \in C(\mathbb{R}) \) satisfies:
\[\lim_{x \to +\infty} f(x) = A, \ \lim_{x \to -\infty} f(x) = B\]
for some constants \( A \) and \( B \). Show that \( \lim_{t \to \infty} u(x,t) = \frac{A + B}{2} \), for each \( x \in \mathbb{R} \).
Hint: Use an integral representation for the solution \( u \). Justify why the representation is valid.

2. If \( f \in W^{1,2}(\Omega) \) with \( \Omega \subset \mathbb{R}^n \) connected and \( Df = 0 \), then prove that \( f \) is constant in \( \Omega \).

3. Let \( B \) be a ball in \( \mathbb{R}^n \), \( f \in C(\partial B) \), and the boundary value problem
\[\begin{cases}
\Delta u = 1 \text{ in } B \\
\frac{\partial u}{\partial v} = f \text{ on } \partial B.
\end{cases}\]
Prove that
\[\begin{align*}
1. & \text{ if } u_1, u_2 \in C^2(\bar{B}) \text{ solve the boundary value problem, then } u_1 - u_2 \text{ is constant in } B; \\
2. & \text{ if there is a solution } u \in C^2(\bar{B}) \text{ to the boundary value problem, then } \int_{\partial B} f(x) \, d\sigma(x) = |B|.\end{align*}\]