Part I. (Do 3 problems)

1. Solve the initial value problem
   \[ u_x + x^2 y u_y = -u, \quad u(0, y) = y^2. \]

2. The Fourier transform is defined by \( \hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i \xi x} \, dx \). Calculate the Fourier transform of the function
   \[ f(x) = \begin{cases} 
   e^{2\pi ibx} & \text{for } |x| \leq a \\
   \frac{1}{\sqrt{a}} & \text{for } |x| > a,
   \end{cases} \]
   and the norm \( \| \hat{f} \|_2 \). The numbers \( a \) and \( b \) are positive.

3. Let \( \Omega \) be a bounded open set in \( \mathbb{R}^n \). Prove the following interpolation inequality:
   \[
   \left( \int_{\Omega} |Du(x)|^2 \, dx \right)^2 \leq \left( \int_{\Omega} u(x)^2 \, dx \right) \left( \int_{\Omega} (\Delta u(x))^2 \, dx \right)
   \]
   for all \( u \in C_0^\infty(\Omega) \); \( Du \) denotes the gradient of \( u \).
   HINT: first prove the following formula valid for all \( v \in C^\infty \): \( \text{div}(v Dv) = v \Delta v + |Dv|^2 \).

4. Let \( u \) be a bounded solution to the heat equation \( u_t - u_{xx} = 0 \) in \( -\infty < x < \infty, t > 0 \) with \( u(x, 0) = f(x) \) with \( f \in L^2(\mathbb{R}) \). Prove that there is a constant \( C > 0 \), independent of \( u \), such that
   \[ \sup_x |u_x(x, t)| \leq C t^{-3/4} \| f \|_2, \quad \text{for all } t > 0. \]

Part II. (Do 2 problems)

1. Let \( \Omega \subset \mathbb{R}^n \) be a bounded regular domain. Prove that if \( u \in W^{1,p}(\Omega) \) and \( v \in W^{1,q}(\Omega) \) with \( 1 \leq p, q \leq \infty \) and \( \frac{1}{p} + \frac{1}{q} = 1 \), then \( uv \in W^{1,1}(\Omega) \).

2. Let \( \Omega \subset \mathbb{R}^n \) be an open bounded domain and let \( u_k \) be a sequence of harmonic functions in \( \Omega \). Suppose that \( u_k \leq u_{k+1} \) for \( k = 1, 2, \cdots \) and there exists \( x_0 \in \Omega \) such that \( u_k(x_0) \) converges. Prove that there exists a harmonic function \( u \) in \( \Omega \) such that \( u_k \to u \) uniformly on compact subsets of \( \Omega \).

3. Let \( \Omega \) be a bounded smooth domain and let \( u \) be smooth in \( \bar{\Omega} \times [0, T] \) solving
   \[ u_{tt} - \Delta u + u^3 = 0 \quad \text{in } \Omega \times [0, T] \]
   \[ u(x, t) = 0 \quad \text{in } \partial\Omega \times [0, T]. \]
   Prove that the energy
   \[ E(t) = \int_{\Omega} \left( u_t^2 + |Du|^2 + \frac{1}{2} u^4 \right) \, dx \]
   is constant in \( [0, T] \).