Ph.D. Comprehensive Examination
Partial Differential Equations
August 2013

Part I. Do three of these problems.

I.1 Suppose $u \in H^1(\mathbb{R}^n)$ and $f \in C^1(\mathbb{R}^n) \cap W^{1,\infty}(R^n)$ (i.e., f is C^1 and it and its first order derivatives are bounded). Show that $fu \in H^1(\mathbb{R}^n)$.

(You may use without proof that there is $\{f_k\}_{k=1}^{\infty} \subset C^\infty(\mathbb{R}^n) \cap W^{1,\infty}(R^n)$ such that $f_k \to f$ in $W^{1,\infty}(\Omega)$ for every open bounded $\Omega \subset \mathbb{R}^n$.)

I.2 Let δ denote the Dirac delta at the origin in $\mathbb{R}$. Show that

$$x^k \frac{d^\ell \delta}{dx^\ell} = (-1)^k \frac{\ell!}{(\ell-k)!} \frac{d^{\ell-k}\delta}{dx^{\ell-k}}$$

for $k, \ell \in \mathbb{N}_0$ with $k \leq \ell$.

I.3 Use the Fourier transform to find a *tempered* (i.e., in $\mathcal{S}'(\mathbb{R})$) fundamental solution of the ordinary differential operator

$$u \mapsto Pu = \frac{du}{dx} + 2u,$$

in other words, find $E \in \mathcal{S}'(\mathbb{R})$ such that $PE = \delta$, where, as in I.2, δ is the Dirac delta.

I.4 Let a, $b \in C^\infty(\mathbb{R}_t \times \mathbb{R}_x)$ be real valued and bounded, define

$$P = \Big(\frac{\partial}{\partial t} + a(t,x)\frac{\partial}{\partial x}\Big)\Big(\frac{\partial}{\partial t} + b(t,x)\frac{\partial}{\partial x}\Big)$$

Show that if $u \in C^2(\mathbb{R}_t \times \mathbb{R}_x)$ satisfies $Pu = 0$ in $\mathbb{R}_t \times \mathbb{R}_x$ and $u = 0$ for $t < 0$, then $u \equiv 0$.

Part II. Do two of these problems.

II.1 Let $B(x_0,\rho) \subset \mathbb{R}^n$ be the ball of center x_0 and radius $\rho > 0$. Suppose $f \in C^\infty(\mathbb{R}^n \backslash 0)$ is positively homogeneous of degree $\mu \in \mathbb{R}$, which means that $f(tx) = t^\mu f(x)$ for all $x \neq 0$ and $t > 0$. Assuming $1 \leq p < \infty$, show:

1. if $\mu + n/p > 0$ then $f \in L^p(B(0,\rho))$;
2. if $\mu + n/p > 1$ then, for any $y \in \mathbb{R}^n$,

$$\frac{f(x+hy) - f(x)}{h} - y \cdot \nabla f(x) \to 0 \text{ in } L^p(B(0,\rho)) \text{ as } h \to 0.$$

Conclude that $f \in W^{1,p}(B(0,\rho))$ if $\mu + n/p > 1$.

II.2 Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with C^2 boundary and outer unit normal field ν. Suppose $u \in C^2(\overline{\Omega} \times [0,\infty))$ is real-valued and satisfies the heat equation

$$\frac{\partial u}{\partial t} - \Delta u = 0 \quad \text{in } \Omega \times (0,\infty), \quad u\big|_{t=0} = u_0 \text{ on } \Omega, \quad \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega \times [0,\infty).$$

1. Show that

$$\int_\Omega |u(x,t)|^2\, dx \leq \int_\Omega |u_0(x)|^2\, dx$$

for all $t \geq 0$.

2. Show that the problem

$$\frac{\partial u}{\partial t} - \Delta u = f \quad \text{in } \Omega \times (0,\infty), \quad u\big|_{t=0} = u_0 \text{ on } \Omega, \quad \frac{\partial u}{\partial \nu} = w \text{ on } \partial\Omega \times [0,\infty).$$

with sufficiently regular data can have only one solution in $C^2(\overline{\Omega} \times [0,\infty))$.

II.3 Let $\Omega \subset \mathbb{R}^n$ be open, and suppose $u \in C(\Omega)$ satisfies the mean value property in Ω,

$$u(x_0) = \frac{1}{|\partial B(x_0,r)|} \int_{\partial B(x_0,r)} u(y)\, d\sigma(y) \quad \text{for all } x_0 \in \Omega \text{ and } r > 0 \text{ s.t. } \overline{B(x_0,r)} \subset \Omega$$

($B(x_0,r)$ was defined in II.1). Show that $\Delta u = 0$ in the weak sense (i.e., that u is harmonic). Note that u is assumed to be continuous, not of class C^2. Hint: Use convolution with $\chi_\varepsilon(x) = \varepsilon^{-n}\chi(x/\varepsilon)$ where χ is a smooth compactly supported function depending only in $|x|$ to reduce to the case where u is smooth.