Part I. Do three of these problems.

I.1 Use the method of characteristics to find a solution of

\[
\begin{align*}
\frac{\partial u}{\partial x} + 2x \frac{\partial u}{\partial y} &= 0 \\
u(0, y) &= \sin(y).
\end{align*}
\]

I.2 Show that \( u(t) = |t|^{1/2} \) \((t \in \mathbb{R})\) is a weak solution of

\[
(t \frac{d}{dt} - \frac{1}{2})u = 0
\]

in \( \mathbb{R} \).

I.3 Define \( K : \mathbb{R}^2 \to \mathbb{R} \) by \( K(x, y) = H(x)H(y) \) where \( H : \mathbb{R} \to \mathbb{R} \) is the Heaviside function. Let \( \Omega = (0, 1) \times (0, 1) \subset \mathbb{R}^2 \). Show that if \( f \in L^p(\Omega) \) \((1 \leq p \leq \infty)\) then

\[
F(x, y) = \int_{\Omega} K(x - \xi, y - \eta)f(\xi, \eta) \, d\xi \, d\eta
\]

belongs to \( W^{1, p}(\Omega) \). Recall that \( H(t) = 1 \) if \( t \geq 0 \), \( H(t) = 0 \) otherwise.

I.4 Let \( \lambda > 0 \), define \( E_\lambda : C(\mathbb{R}^2) \to \mathbb{C} \) by

\[
E_\lambda(f) = \int_0^{2\pi} f(\lambda \cos \theta, \lambda \sin \theta) \, d\theta.
\]

For \( x \in \mathbb{R}^2 \) let \( w_x(\xi) = e^{ix \cdot \xi} \), so \( w_x \in C(\mathbb{R}^2) \) for each \( x \); here \( i = \sqrt{-1} \). Define \( u : \mathbb{R}^2 \to \mathbb{C} \) by

\[
u(x) = E_\lambda(w_x).
\]

a) Give a detailed argument as to why \( u \in C^2 \).

b) Let \( \Delta \) be the standard Laplacian in \( \mathbb{R}^2 \). Show that \( \Delta u = -\lambda^2 u \).
Part II. Do two of these problems.

II.1 Let $\Omega \subset \mathbb{R}^3$ be a bounded open set with $C^1$ boundary, let $V \in C^2(\Omega)$ be real-valued, and suppose that $\psi \in C^2(\mathbb{R} \times \Omega)$ satisfies
\[
\frac{1}{i} \frac{\partial \psi}{\partial t} = - \Delta \psi + V \psi \quad \text{in } \mathbb{R} \times \Omega, \quad \psi = 0 \text{ on } \mathbb{R} \times \partial \Omega.
\]
Show that
\[
\int_{\Omega} |\psi(t, x)|^2 dx,
\]
a function of $t \in \mathbb{R}$, is constant. Here $\Delta$ is the standard Laplacian in the $x$ variables and $i = \sqrt{-1}$. Hints: Differentiate in $t$, pay attention to real and imaginary parts, use the divergence theorem at some point; remember to justify each step.

II.2 Suppose $u$ is a $C^2$ function on $\mathbb{R}^2 = \mathbb{R}_t \times \mathbb{R}_x$ such that
\[
\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0.
\]
Show that if there is $r > 0$ such that $u(t, x) = 0$ for $|x| > r$ and $t \in \mathbb{R}$, then $u \equiv 0$.

II.3 Let $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$ and let $\Omega \subset \mathbb{R}$ be the interval $(\alpha, \beta)$. Further, let $a, b \in C(\overline{\Omega})$ be real-valued with $a$ strictly positive, and define
\[
Lu = -a \frac{d^2 u}{dx^2} + b \frac{du}{dx} \quad \text{on } \Omega.
\]
Suppose $u \in C^2(\Omega) \cap C(\overline{\Omega})$. Show:

If $Lu \leq 0$ and $\max_{\Omega} u$ is attained in $\Omega$, then $u$ is constant.

Hint: Suppose the maximum $M$ is attained at a point $x_0 \in \Omega$ and $u \neq M$. Consider the auxiliary function $z(x) = \exp(\lambda(x - x_0)) - 1$, and choose $\lambda$ large enough so that $Lz < 0$ in $\Omega$. Use $w(x) = u(x) + \varepsilon z(x)$, with $\varepsilon > 0$ sufficiently small, to obtain a contradiction.

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