Part I. (Do 3 problems)

1. Solve the Cauchy problem

\[-x u_x + u_y = -u + 1, \quad -\infty < x, y < \infty,\]
\[u(x,0) = \sin x, \quad -\infty < x < \infty.\]

2. The Fourier transform is defined by \( \hat{f}(x) = \int_{\mathbb{R}^n} f(t) e^{-2\pi i x \cdot t} \, dt \). Recall that if \( \alpha > 0 \) and \( f(t) = e^{-2\pi \alpha |t|} \), then \( \hat{f}(x) = C_n \frac{\alpha}{\alpha^2 + |x|^2} \). Prove the following formula valid for \( \alpha, \beta > 0 \):

\[
C_n \int_{\mathbb{R}^n} \frac{1}{(\alpha^2 + |x|^2)^{n+1}/2} \cdot \frac{1}{(\beta^2 + |x|^2)^{n+1}/2} \, dx = \frac{1}{\alpha \beta (\alpha + \beta)^n}.
\]

3. Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) with \( C^1 \) boundary. If \( \lambda \leq 0 \), show that the only solution \( u \in C^2(\bar{\Omega}) \) to

\[\Delta u + \lambda u = 0 \text{ in } \Omega\]
\[u = 0 \text{ on } \partial \Omega\]

is \( u \equiv 0 \).

HINT: use the divergence theorem.

4. Let \( \phi \in C^2(0, +\infty) \) and suppose \( u(x,t) = \phi(x/t) \) solves the one dimensional wave equation \( u_{tt} - u_{xx} = 0 \) for \( x > 0, t > 0 \). Show that there exist constants \( A, B \) such that

\[u(x,t) = A \ln \left( \frac{t + x}{t - x} \right) + B\]

for \( 0 < x < t \).
Part II. (Do 2 problems)

1. Let \( f(x) = |x_1|^3 \) in the unit ball \( B_1(0) \subset \mathbb{R}^n \). Prove that \( f \) has weak derivatives \( D^\alpha f \) for all \( |\alpha| = 1 \) with \( D_1 f(x) = 3 \text{sign}(x_1) x_1^2 \) and \( D_j f(x) = 0 \) for \( j = 2, \ldots, n \).

2. Let \( k \) be an odd positive integer and let \( \Omega \subset \mathbb{R}^n \) be a bounded \( C^1 \) domain. Prove that if \( u(x, t) \in C^2(\bar{\Omega} \times [0, T]) \) is a solution to

\[
\begin{align*}
    u_t &= \Delta u - u^k \quad \text{in} \quad \Omega \times [0, T], \\
    u(x, t) &= 0 \quad \text{on} \quad \partial \Omega \times [0, T], \quad \text{and} \quad u(x, 0) = 0 \quad \text{on} \quad \Omega,
\end{align*}
\]

then \( u \equiv 0 \) on \( \Omega \times [0, T] \).

HINT: multiply the equation by \( u \) and integrate.

3. Let \( D(r) = \{(x, y) : x^2 + y^2 \leq r^2\} \), and let \( u(x, y) \) be a non constant harmonic function in the unit disk \( D(1) \). Let \( \phi(r) = \max_{(x, y) \in \partial D(r)} u(x, y) \) for \( 0 \leq r \leq 1 \).

Prove that

(a) \( u \) is non constant on each disk \( D(r) \) for all \( 0 < r < 1 \).

HINT: suppose \( u \) is constant in \( D(r) \) for some \( r < 1 \), and use that \( u \) is real analytic in \( D(1) \).

(b) \( \phi(r) \) is non decreasing in \([0, 1] \).

(c) \( \phi(r) \) is strictly increasing in \([0, 1] \).