PDEs Ph.D. Qualifying Exam
Temple University
August 25, 2011

Part I. (Do 3 problems)

1. Solve the Cauchy problem

$$-xu_x + u_y = -u + 1, \ -\infty < x, y < \infty,$$
$$u(x, 0) = \sin x, \ -\infty < x < \infty.$$

2. The Fourier transform is defined by $\hat{f}(x) = \int_{\mathbb{R}^n} f(t)\, e^{-2\pi i\, x\cdot t}\, dt$. Recall that if $\alpha > 0$ and $f(t) = e^{-2\pi\alpha|t|}$, then $\hat{f}(x) = C_n \dfrac{\alpha}{(\alpha^2 + |x|^2)^{(n+1)/2}}$. Prove the following formula valid for $\alpha, \beta > 0$:

$$C_n \int_{\mathbb{R}^n} \frac{1}{(\alpha^2 + |x|^2)^{(n+1)/2}} \frac{1}{(\beta^2 + |x|^2)^{(n+1)/2}}\, dx = \frac{1}{\alpha\beta(\alpha+\beta)^n}.$$

3. Let Ω be a bounded domain in $\mathbb{R}^n$ with C^1 boundary. If $\lambda \le 0$, show that the only solution $u \in C^2(\bar{\Omega})$ to

$$\Delta u + \lambda u = 0 \text{ in } \Omega$$
$$u = 0 \text{ on } \partial\Omega$$

is $u \equiv 0$.

HINT: use the divergence theorem.

4. Let $\phi \in C^2(0, +\infty)$ and suppose $u(x, t) = \phi(x/t)$ solves the one dimensional wave equation $u_{tt} - u_{xx} = 0$ for $x > 0, t > 0$. Show that there exist constants A, B such that

$$u(x, t) = A \ln\left(\frac{t + x}{t - x}\right) + B$$

for $0 < x < t$.

Part II. (Do 2 problems)

1. Let $f(x) = |x_1|^3$ in the unit ball $B_1(0) \subset \mathbb{R}^n$. Prove that f has weak derivatives $D^\alpha f$ for all $|\alpha| = 1$ with $D_1 f(x) = 3\,\mathrm{sign}(x_1)\,x_1^2$ and $D_j f(x) = 0$ for $j = 2, \cdots, n$.

2. Let k be an odd positive integer and let $\Omega \subset \mathbb{R}^n$ be a bounded C^1 domain. Prove that if $u(x,t) \in C^2(\bar{\Omega} \times [0,T])$ is a solution to

$$u_t = \Delta u - u^k \quad \text{in} \quad \Omega \times [0,T],$$
$$u(x,t) = 0 \text{ on } \partial\Omega \times [0,T], \text{ and } u(x,0) = 0 \text{ on } \Omega,$$

   then $u \equiv 0$ on $\Omega \times [0,T]$.

   HINT: multiply the equation by u and integrate.

3. Let $D(r) = \{(x,y) : x^2 + y^2 \leq r^2\}$, and let $u(x,y)$ be a non constant harmonic function in the unit disk $D(1)$. Let $\phi(r) = \max\limits_{(x,y)\in\partial D(r)} u(x,y)$ for $0 \leq r \leq 1$.

   Prove that

   (a) u is non constant on each disk $D(r)$ for all $0 < r < 1$.
   HINT: suppose u is constant in $D(r)$ for some $r < 1$, and use that u is real analytic in $D(1)$.

   (b) $\phi(r)$ is non decreasing in $[0,1]$.

   (c) $\phi(r)$ is strictly increasing in $[0,1]$.