Numerical Analysis Qualifying Written Exam (August 2022)

Part I: do 3 of 4

1. Construct a fixed point iteration to find the value of $\sqrt{a}$, $a > 0$.
(a) Prove whether your iteration converges and explain carefully.
(b) Find the convergence rate of your iteration.

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2. Looking at floating point arithmetic (FPA),
(a) explain machine precision and the fundamental axiom of FPA:

$$\forall x, y \in F, \exists \varepsilon, |\varepsilon| \leq \varepsilon_{mach} : x \otimes y = (x \times y)(1 + \varepsilon),$$

where $\otimes$ denotes a floating point operation and $\times$ the exact operation.
(b) show that FPA is not associative,
(c) based on (b) give two examples of computing $\mathrm{fl}\left(\sum_{i=1}^{n} x_i y_i\right)$ and explain the (potential) advantages of one over the other.

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3. Given the ODE Initial Value Problem (IVP) $u'(t) = f(u(t))$, $u(0) = u_0$, $t \in [0, T]$:
(a) Combine the Forward Euler and Backward Euler method to construct the trapezoidal method.
(b) For the three methods in (a) find the local truncation error and explain your findings.
(c) Define *consistency*, *zero-stability*, and *absolute stability*. Under which conditions is a numerical ODE-solving scheme convergent. Explain.

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4. Consider the method $U^{n+1} = U^n + \frac{1}{2}k\lambda(U^{n+1} + U^n)$ for solving the problem $u'(t) = \lambda u(t)$, $u(0) = u_0$, $t \in [0, T]$, $\lambda \in \mathbb{C}$.
(a) Find the region of absolute stability (RAS) of above method and compare to the RAS of Forward Euler.
(b) Now suppose $u = \begin{bmatrix} v \\ w \end{bmatrix}$ and $\begin{bmatrix} v \\ w \end{bmatrix}' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix}$. Given both methods studied above, carefully analyze which of the two methods you would use here.

Part II: do 2 of 3

1. Suppose you have data (x_i, y_i), $i = 0 \dots n$. Using the basis $\{1, x, x^2, \dots, x^n\}$ to span the polynomial space $\Pi_n = \{p(x) : \text{polynomials of degree} \leq n\}$:
(a) Define an interpolating polynomial $p(x)$ to the data.
(b) Show that finding $p(x)$ corresponds to solving a system of linear equations.
(c) Prove that $p(x)$ in (b) is unique.
(d) Explain why using above basis to construct an interpolating polynomial, when using a computer, is generally not ideal.

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2. Consider the integral $\int_0^1 f(x)dx$, with $f(x) = \frac{1}{(x-2)^2}$.
(a) Explain the interpolating polynomial

$$p(t) = f(0)[(t-1)^2 + 2t(t-1)^2] + f'(0)t(t-1)^2 + f(1)[t^2 - 2t^2(t-1)] + f'(1)t^2(t-1)$$

using your knowledge about Hermite interpolation.
(b) Use the Euler-Maclaurin method to solve above integral. Make use of part (a) and explain your work.

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3. Consider one-step methods for solving a linear Initial Value Problem $u'(t) = \lambda u(t)$, $u(0) = u_0$, $t \in [0, T]$, $\lambda \in \mathbb{C}$.
(a) Define *L-stability*.
(b) Find the growth factor $R(z) = \frac{U^{n+1}}{U^n}$, $z = k\lambda$, k being the time step, for the Backward Euler method and the method defined by $U^{n+1} = U^n + \frac{k\lambda}{2}(U^{n+1} + U^n)$. Sketch the graph of $R(z)$ with respect to k for both methods. Explain the consequences for both methods based on these graphs.
(c) How are your findings in (b) connected to L-stability in (a)? Explain.