Numerical Analysis Qualifying Written Exam (August 2021)

Part I: do 3 of 4

1. Consider a floating point system $F(\beta, t, m, M)$.
(a) Show that addition in these system is not associative.
(b) Define when an algorithm is backward stable.
(c) Show that the addition of two floating point numbers is a backward stable operation.

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2. Consider a fixed point problem $x = F(x)$, and the fixed point iteration $x_k = F(x_{k-1})$.
(a) State under which conditions the fixed point iteration converges,
(b) Prove convergence under the conditions in (a).
(c) Explain the concepts of linear, superlinear, and quadratic convergence.
(d) What is the convergence rate of the fixed point iteration? Show in detail why this is so.

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3. Given the ODE Initial Value Problem (IVP) $u'(t) = f(u(t))$, $u(0) = u_0$, $t \in [0, T]$:
(a) Define the Forward (FE) and Backward Euler (BE) methods.
(b) Determine whether these methods are consistent. Does this imply convergence? Explain!
(c) What is the order of accuracy of these methods.?
(d) Explain the difference between zero-stability and absolute stability.
(e) Derive the Region of Absolute Stability (RAS) for FE and BE.
(f) What is the key difference between the RAS in **(e)**? What does this tell you about the difference between explicit and implicit methods?

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4. Consider the Butcher tableau

$$\begin{array}{c|ccc} 0 & 0 & & \\ 1/2 & 1/4 & 1/4 & \\ 1 & 1/3 & 1/3 & 1/3 \\ \hline & 1/3 & 1/3 & 1/3 \end{array}$$

(a) Write out the procedure defined by above tableau.
(b) Define A-stability and L-stability.
(c) Prove that this method is L-stable.

Part II: do 2 of 3

1. Consider an interpolation problem with the $n+1$ points $x_0, x_1, \ldots, x_n$, with abcisas $f_0, \ldots, f_n$. Let the interpolation polynomial of degree n be $p(x)$. Suppose that once you have $p(x)$ you have a new piece of data, namely f'_j for some $0 \le j \le n$. One is seeking an new interpolating polynomial $q(x)$ so that $q(x_i) = f_i$, $i = 0, \ldots, n$ and $q'(x_j) = f'_j$. Can you use the polynomial already computed $p(x)$ and just compute a new polynomial $r(x)$ so that $q(x) = p(x) + r(x)$? Give as much detail as possible. What is the degree of $r(x)$? Is this polynomial unique?

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2. Choose n (the number of points) and use a Gauss-Chebyshev formula to evaluate (exactly) the following integrals

$$(i) \int_{-1}^{1} \frac{x^3}{\sqrt{1-x^2}} dx \qquad\qquad (ii) \int_{0}^{1} \frac{x^2}{\sqrt{x(1-x)}} dx$$

Describe every step of your logic and computations.

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3. Give and example of a stiff ODE Initial Value Problem (IVP). Then:
(a) Illustrate and explain what happens when you perturb the initial value.
(b) Choose a method to numerically solve your ODE IVP and explain your choice.
(c) Why Runge-Kutta-Chebyshev (RKC) methods are potentially a good choice for stiff problems? Explain the RKC methods to argue your point.