Part I. Solve three of the following problems.

I.1 Let $X$ be the space obtained from identifying sides of the polygon $\mathcal{P}$ in Figure 1 as indicated. Compute $\pi_1(X)$ and the homology groups $H_*(X;\mathbb{Z})$.

I.2 Consider the function

$$f(x, y, z) = xy + yz + zx + x + y + z$$

from $\mathbb{R}^3$ to $\mathbb{R}$. Classify all the $t \in \mathbb{R}$ such that $M_t = f^{-1}(t)$ is a smooth 2-dimensional submanifold of $\mathbb{R}^3$.

I.3 Let $K$ be a Klein bottle and $X$ be the wedge product $\mathbb{RP}^2 \vee \mathbb{RP}^2$ of two real projective planes. If $f : X \to K$ is a continuous mapping, prove that $f$ is homotopic to a constant map.

I.4 Suppose that $X$ and $Y$ are smooth manifolds and that $f : X \to Y$ is a smooth map with differential $(Df_x) : T_xX \to T_{f(x)}Y$.

(a) Let $V$ be a smooth vector field on $Y$, with $V_y \in T_yY$ the vector in $V$ at $y$. Say that $V$ is tangent to $f$ if

$$V_{f(x)} \in Df(T_xX) \subseteq T_{f(x)}Y$$

for all $x$ in $X$. If $f$ is an immersion and $V$ is tangent to $f$, prove that $(Df)^{-1}(V_{f(x)})$ determines a well-defined smooth vector field $f^*(V)$ on $X$. If $V$ is nowhere-vanishing, prove that $f^*(V)$ is also nowhere-vanishing.

(b) Give an example of $X, Y, f$, and a nowhere-vanishing $V$ satisfying all the assumptions of part (a).
Part II. Solve two of the following problems.

II.1 Let $R_0$ denote the graph with two vertices and one edge between them; see Figure 2. For $n \geq 1$, the rose with $n$ petals $R_n$ is the unique finite graph with 1 vertex and $n$ edges; see Figure 3.

Figure 2: The graph $R_0$.

Figure 3: The graph $R_4$.

(a) Prove that any finite connected graph $G$ is homotopy equivalent to $R_n$ for some $n \geq 0$.

(b) Let $G$ be a finite connected graph, $V$ denote the number of vertices of $G$, and $E$ the number of edges. Give a formula for the $n$ in part (a) in terms of $V$ and $E$.

(c) Let $G$ be as above and $\mathcal{H} \to G$ be a connected covering of degree $d$. Show that $\pi_1(\mathcal{H})$ is a free group of rank $m \geq 0$, where $m$ is an explicit function of: $d$, the number $V$ of vertices of $G$, and the number $E$ of edges of $G$.

II.2 Let $M$ be a smooth compact oriented manifold of dimension $n$ without boundary and $\Omega^q(M)$ be the space of smooth exterior forms of degree $q$ on $M$. Prove that the assignment

$$I(\omega) := \int_M \omega$$

induces an $\mathbb{R}$-linear map

$$I : \Omega^n(M)/d(\Omega^{n-1}(M)) \to \mathbb{R},$$

where $d$ is the exterior derivative. Use a partition of unity to prove that the map $I$ is onto.
II.3 Let $X = S^1$ with coordinate

$$\theta \mapsto z = \cos \theta + i \sin \theta \in \mathbb{C}$$

and $Y = X \times X$.

(a) Show that the map $f_n(z) = (z, z^n)$ defines a smooth immersion of $X$ into $Y$ for all $n \in \mathbb{Z}$.

(b) Give an example of a smooth map $h : Y \to X$ such that $h \circ f_1$ is a submersion at every $z \in X$.

(c) Give an example of a smooth map $h : Y \to X$ such that $h \circ f_1$ fails to be a submersion for every $z \in X$. 