Comprehensive Examination in Geometry & Topology
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Part I. Solve three of the following problems.

I.1 Let $x_1, \ldots, x_n$ be distinct points on the 2-sphere S^2, and define

$$X_n = S^2 \setminus \{x_1, \ldots, x_n\}.$$

a. Compute $\pi_1(X_n)$, $n \geq 0$.

b. Draw all the equivalence classes of 2-sheeted covers of X_3 relative to your base point.

I.2 Show that

$$\omega(x, y) = \cos(2\pi x)\, dx + \sin(2\pi y)\, dy$$

is a well-defined smooth 1-form on $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. Is it closed? Is it exact?

I.3 Let Σ_g be an orientable closed surface of genus g. Prove that Σ_g admits a nowhere vanishing vector field if and only if $g = 1$.

I.4 For all $g \geq 1$, prove that any continuous map from the real projective plane $\mathbb{RP}^2$ to a closed orientable surface Σ_g of genus g is homotopic to a constant map. *Hint: Recall that the universal cover of Σ_g (for $g \geq 1$) is contractible.*

Part II. Solve two of the following problems.

II.1 Let M be a manifold without boundary and $v : M \to TM$ be a smooth vector field with finitely many zeros. For every zero $p_0 \in M$ of v, assume that the matrix with (i, j) entry

$$\frac{\partial v^i(\vec{x})}{\partial x^j}\Big|_{p_0}$$

is nondegenerate. Prove that the map $v : M \to TM$ is traverse to the zero section $M \subset TM$. Here $x^1, \dots, x^n$ are local coordinates in a neighborhood of p_0 and $v^i(\vec{x})$ are the components of the vector field v.

II.2 Prove that the formula

$$f_n(z_0 : z_1) := (z_0^n : z_1^n)$$

defines a smooth map from $\mathbb{CP}^1$ to $\mathbb{CP}^1$ for every non-zero integer n. Find the degree of f_n. Here z_0, z_1 are homogeneous coordinates on $\mathbb{CP}^1$.

II.3 Let X be the 2-torus and Y a genus 2 surface. Let $a \subset X$ and $b \subset Y$ be the pictured curves, and glue X to Y via a degree 2 map $a \to b$ to obtain a new space Z. Compute $\pi_1(Z)$ and $H_1(Z, \mathbb{Z})$.