Part I. Solve three of the following problems.

I.1 Prove that 
\[ O(n) = \{ A \in M_n(\mathbb{R}) : \; A^t A = \text{Id} \} \]
is a smooth submanifold of the space \( M_n(\mathbb{R}) = \mathbb{R}^{n^2} \) of \( n \times n \) matrices. Here, \( A^t \) is the transpose of the matrix \( A \). Describe a natural identification of the tangent space to \( M_n(\mathbb{R}) \) at the identity with \( M_n(\mathbb{R}) \), and describe the tangent space to \( O(n) \) at the identity as a subspace of \( M_n(\mathbb{R}) \).

I.2 Give a careful definition of a normal covering map. Also, give

- an example of a normal covering map \( p \) that is not a homeomorphism, and
- an example of a non-normal covering map \( p \).

For both examples, prove that the corresponding covering map is normal (resp. non-normal). 

*Hint: there is a non-normal covering map with the base being the Klein bottle.*

I.3 Consider the circle \( S^1 \) as the unit circle in \( \mathbb{R}^2 \) with coordinate \( \theta \), then give the torus \( T = S^1 \times S^1 \) into \( \mathbb{R}^4 \) coordinates \((\theta_1, \theta_2)\). Define differential forms
\[ \eta_i = d\theta_i \quad i = 1, 2. \]

(a) Given \((a, b), (c, d) \in \mathbb{R}^2\), calculate
\[ (a\eta_1 + b\eta_2) \wedge (c\eta_1 + d\eta_2). \]

(b) Calculate
\[ \int_T \eta_1 \wedge \eta_2. \]

(c) Prove that \( a\eta_1 + b\eta_2 \) is closed, but not exact, for every \((a, b) \in \mathbb{R}^2\) except \((0, 0)\).

I.4 Let \( M \) be a closed 4-manifold and let \( \mathbb{CP}^2 \) be the complex projective plane. Compute the homology groups \( H_*(X; \mathbb{Z}) \) of the blowup \( X = M \# \mathbb{CP}^2 \) of \( M \) at a point (i.e., the connect sum of \( M \) and \( \mathbb{CP}^2 \)) in terms of the homology groups of \( M \).
Part II. Solve two of the following problems.

II.1 Let $T = \mathbb{R}^2/\mathbb{Z}^2$ be a 2-torus.

(a) Let $L$ be a line of slope $(p, q)$ in $\mathbb{R}^2$, with $p$ and $q$ integers (not both zero). Prove that $L$ projects to a homotopically nontrivial closed loop $\sigma_{p,q}$ on $T$.

(b) Let $T_1$ and $T_2$ be two copies of $T$, and let $X$ be the space given by gluing $T_1$ to $T_2$ by gluing a $(p_1, q_1)$ curve on $T_1$ to a $(p_2, q_2)$ curve on $T_2$. Compute $\pi_1(X)$. You do not need to prove that $\pi_1(T) \cong \mathbb{Z} \times \mathbb{Z}$.

II.2 State the classification of closed, connected 2-dimensional manifolds. For each integer $k \geq 1$, identify all such manifolds $M$ with $H_k(M; \mathbb{Z}) = \{0\}$. Also identify all such $M$ with $H_k(M; \mathbb{Q}) = \{0\}$.

II.3 Let $X$ be a closed manifold and $Y$ an embedded closed submanifold with Euler characteristic zero. Show that $X$ and $X \setminus Y$ have the same Euler characteristic.