Part I. Solve three of the following problems.

I.1 Let $K$ denote the Klein bottle.

(a) Prove that $K$ contains an embedded Mobüs band $M$.

(b) Let $X$ be the space obtained by gluing two copies of $K$ together along $M$. Compute $\pi_1(X)$.

(c) Let $X$ be the space defined in (b). Compute the homology groups $H_\ast(X)$ using the Mayer–Vietoris sequence.

I.2 Show that the set of rank 1 matrices is a submanifold of the space of real $2 \times 2$ matrices. (Recall that the rank of linear map $\mathbb{R}^m \to \mathbb{R}^n$ is the dimension of its image.) What is its dimension?

I.3 Let $z_1, z_2$ be two distinct points on $\mathbb{RP}^2$ and $X$ be the space obtained by identifying $z_1$ with $z_2$. Put an explicit $\Delta$-complex structure on $X$ and use it to compute the homology groups $H_\Delta(X)$.

I.4 For $n \geq 1$, let $\alpha$ be a closed $n$-form on the $2n$-dimensional sphere $S^{2n}$. Show that the $2n$-form $\alpha \wedge \alpha$ is zero at some point on $S^{2n}$. (Hint: You may use the fact that $H^n_{dR}(S^{2n}) = 0$.)
Part II. Solve two of the following problems.

II.1 Let $M$ and $N$ be closed (compact, without boundary) smooth manifolds of dimension $n$ and let $f: M \to N$ and $g: N \to M$ be smooth maps. Also suppose that $N$ is connected. Show that if $g \circ f$ is a diffeomorphism, then so are $f$ and $g$. What can happen if $N$ is not connected?

II.2 Let $G$ be a finite group and $R_2$ be the rose with 2 petals.

(a) Prove that there is a pair of normal finite covers $\alpha: Z \to Y$ and $\beta: Y \to R_2$ so that $\alpha$ has $G$ as its group of Deck transformations.

(b) Give suitable $Y$, $Z$, $\alpha$, and $\beta$ for $G = (\mathbb{Z}/2)^3$, the direct product of three cyclic groups of order two.

II.3 Let $X$ be a topological space and $Y \subset X$ a subspace.

(a) Define a retraction of $X$ onto $Y$.

(b) Prove that if $X$ is simply connected and $X$ retracts onto $Y$, then $Y$ is simply connected.

(c) Proof or counterexample: Is the converse true? That is, if $Y$ is simply connected and $X$ retracts onto $Y$, is $X$ necessarily simply connected?

(d) Let $M$ be a compact orientable smooth manifold with boundary $\partial M$. Prove that there is no retraction of $M$ onto $\partial M$. 