Comprehensive Examination in Geometry & Topology
Department of Mathematics, Temple University

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Part I. Solve three of the following problems.

I.1 Let K denote the Klein bottle.

(a) Prove that K contains an embedded Mobüs band M.

(b) Let X be the space obtained by gluing two copies of K together along M. Compute $\pi_1(X)$.

(c) Let X be the space defined in (b). Compute the homology groups $H_*(X)$ using the Mayer–Vietoris sequence.

I.2 Show that the set of rank 1 matrices is a submanifold of the space of real 2×2 matrices. (Recall that the rank of linear map $\mathbb{R}^m \to \mathbb{R}^n$ is the dimension of its image.) What is its dimension?

I.3 Let z_1, z_2 be two distinct points on $\mathbb{RP}^2$ and X be the space obtained by identifying z_1 with z_2. Put an explicit Δ-complex structure on X and use it to compute the homology groups $H_*^{\Delta}(X)$.

I.4 For $n \geq 1$, let α be a closed n-form on the $2n$-dimensional sphere S^{2n}. Show that the $2n$-form $\alpha \wedge \alpha$ is zero at some point on S^{2n}. (Hint: You may use the fact that $H^n_{\mathrm{dR}}(S^{2n}) = 0$.)

Part II. Solve two of the following problems.

II.1 Let M and N be closed (compact, without boundary) smooth manifolds of dimension n and let $f\colon M \to N$ and $g\colon N \to M$ be smooth maps. Also suppose that N is connected.

Show that if $g \circ f$ is a diffeomorphism, then so are f and g. What can happen if N is not connected?

II.2 Let G be a finite group and R_2 be the rose with 2 petals.

(a) Prove that there is a pair of normal finite covers $\alpha : Z \to Y$ and $\beta : Y \to R_2$ so that α has G as its group of Deck transformations.

(b) Give suitable Y, Z, α, and β for $G = (\mathbb{Z}/2)^3$, the direct product of three cyclic groups of order two.

II.3 Let X be a topological space and $Y \subset X$ a subspace.

(a) Define a retraction of X onto Y.

(b) Prove that if X is simply connected and X retracts onto Y, then Y is simply connected.

(c) Proof or counterexample: Is the converse true? That is, if Y is simply connected and X retracts onto Y, is X necessarily simply connected?

(d) Let M be a compact orientable smooth manifold with boundary ∂M. Prove that there is no retraction of M onto ∂M.