Part I: Do three of the following problems.

1. Suppose \( f: M \rightarrow N \) is a smooth map between smooth manifolds and \( S \subset N \) is a smooth submanifold. Suppose that for every \( p \in f^{-1}(S) \), the vector spaces \( f_*(T_pM) \) and \( T_{f(p)}S \) span \( T_{f(p)}N \). Prove that \( f^{-1}(S) \) is a smooth submanifold of \( M \), and compute its dimension.

2. Let \( X = T^2 \vee S^1 \). Compute \( \pi_1(X) \), draw a picture of the universal cover \( \tilde{X} \), and explain how \( \pi_1(X) \) acts on \( \tilde{X} \).

3. Let \( X \) be a connected \( CW \) complex such that \( H_1(X) \cong \mathbb{Z}/3 \).
   (a) Does \( X \) have a connected 2–fold cover? Prove your answer.
   (b) Does \( X \) have a connected 3–fold cover? Prove your answer.

4. Let \( M \) be a non-orientable manifold and \( B^k \) a ball. Prove that \( M \times B^k \) is non-orientable.

Part II: Do two of the following problems.

1. Let \( M \) be a closed, orientable, smooth manifold. Let \( S \subset M \) be an orientable submanifold of codimension 1. Construct a smooth flow \( \varphi_t: M \rightarrow M \) that takes \( S \) off itself. That is, construct \( \varphi_t \) and show there is an \( \epsilon > 0 \) such that \( \varphi_t(S) \cap S = \emptyset \) for \( t \in (0, \epsilon) \).

2. This problem is about the reduced homology groups of \( \mathbb{R}P^n \).
   (a) Prove that
   \[
   \tilde{H}_n(\mathbb{R}P^n) = \begin{cases} 
   0, & n \text{ is even} \\
   \mathbb{Z}, & n \text{ is odd}.
   \end{cases}
   
   (b) Use the long exact sequence of the pair \( (\mathbb{R}P^n, \mathbb{R}P^{n-1}) \) to show that
   \[
   \tilde{H}_{n-1}(\mathbb{R}P^n) = \begin{cases} 
   \mathbb{Z}/2, & n \text{ is even} \\
   0, & n \text{ is odd}.
   \end{cases}
   
   (c) Use induction to compute \( \tilde{H}_k(\mathbb{R}P^n) \) for all \( k \). You may assume the answer to parts (a), (b) in addition to standard facts about homology groups of standard spaces.
3. Consider the torus $T = S^1 \times S^1 \subset \mathbb{R}^2 \times \mathbb{R}^2$ given by

$$T = \{(w, x, y, z) \in \mathbb{R}^2 \times \mathbb{R}^2 : w^2 + x^2 = 1 = y^2 + z^2\},$$

Consider the 2-form $\omega = xyz \, dw \wedge dy$, restricted to $T$.

(a) Is $\omega$ closed? Justify your answer.

(b) Is $\omega$ exact? Justify your answer. *Hint:* integrating $\omega$ over $T$ is helpful.