Comprehensive Examination in Geometry & Topology
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Part I. Solve three of the following problems.

I.1 Prove that a closed surface admits a connected normal cover of every degree $d \geq 1$ if and only if it has nonpositive Euler characteristic.

I.2 Let M be a smooth manifold of dimension n and let $E \to M$ be a vector bundle over M. Prove that the zero section $s : M \to E$, defined locally as $x \mapsto (x, 0)$, is a smooth immersion.

I.3 Let M be a smooth manifold and $f : M \to \mathbb{R}$ a continuous function that is everywhere positive. Use partitions of unity to prove that there is a smooth function $g : M \to \mathbb{R}$ such that $0 < g(x) < f(x)$ for every x.

I.4 Construct a cell complex X whose integral homology groups are as follows:

$$H_i(X) = \begin{cases} \mathbb{Z}^2 & i = 0 \\ \mathbb{Z} \oplus \mathbb{Z}/3 & i = 1 \\ 0 & i = 2 \\ \mathbb{Z} & i = 3 \\ 0 & i > 3 \end{cases}$$

Part II. Solve two of the following problems.

II.1 For $i = 1, 2$, let $T_i = \mathbb{R}^2/\mathbb{Z}^2$ be a 2-torus and $\{a_i, b_i\}$ the standard generators for $\pi_1(T_i)$. Consider the identification space

$$Y = T_1 \sqcup T_2/\sim$$

where $\sim$ identifies b_1 with $a_2^3 b_2^5$ (considered as embedded loops on T_1 and T_2).

(a) Compute $\pi_1(Y)$.

(b) Compute $H_*(Y)$.

(c) Prove that Y is not homotopy equivalent to a closed oriented surface of genus g for any $g \geq 0$.

II.2 Fix $n \geq 1$.

(a) Prove that the group $\mathrm{SL}_n(\mathbb{C})$ of $n \times n$ complex matrices of determinant one is a manifold of (real) dimension $2n^2 - 2$.

(b) Let Id_n be the $n \times n$ identity matrix, and let M^* denote the complex conjugate transpose of a matrix $M \in \mathrm{SL}_n(\mathbb{C})$. Prove that the special unitary group

$$\mathrm{SU}(n) = \{M \in \mathrm{SL}_n(\mathbb{C}) \ : \ M^*M = \mathrm{Id}_n\}$$

is a submanifold of $\mathrm{SL}_n(\mathbb{C})$ and calculate its dimension.

II.3 Let $\gamma \to \mathbb{R}^2$ be the oriented closed curve shown below. The points of self-intersection are $(0, 0)$, $(2, 1)$, and $(4, 2)$. Compute $\int_\gamma y\, dx$.