Comprehensive Examination in Geometry & Topology
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Part I. Solve three of the following problems.

I.1 Prove that the equations

$$x_1^2 - x_2^2 - x_3^2 + x_4^2 - x_3 = 0$$
$$2x_1x_2 - 2x_3x_4 - x_4 = 0$$

define a submanifold of $\mathbb{R}^4$ and find its dimension.

I.2 Consider S^n with the standard atlas $\{U_N, U_S\}$ coming from the stereographic projections from the north and south poles $(0, \dots, 0, \pm 1)$. Let $x^1, \dots, x^n$ be coordinates on U_N and $y^1, \dots, y^n$ be coordinates on U_S. Prove that the formulas

$$v\big|_{U_N} = \sum_{i=1}^{n} x^i \partial_{x^i}$$
$$v\big|_{U_S} = -\sum_{i=1}^{n} y^i \partial_{y^i}$$

define a smooth vector field on S^n.

I.3 Let Σ_g denote a closed orientable surface of genus $g \geq 0$. Prove using homology groups that the wedge product $\Sigma_{g_1} \vee \Sigma_{g_2}$ is never homotopy equivalent to the connect sum $\Sigma_{g_1} \# \Sigma_{g_2}$.

I.4 Prove that a closed orientable genus g surface admits an irregular (i.e., not normal) cover if and only if $g \geq 2$.

Part II. Solve two of the following problems.

II.1 Let I_n (resp. 0_n) be the identity matrix (resp. the zero matrix) of size $n \times n$ and

$$J := \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}.$$

a) Prove that the subset

$$\mathrm{Sp}(2n) := \left\{ A \in \mathrm{M}_{2n}(\mathbb{R}) \ : \ A^t J A = J \right\} \subset \mathrm{M}_{2n}(\mathbb{R})$$

is a submanifold of $\mathrm{M}_{2n}(\mathbb{R}) = \mathbb{R}^{(4n^2)}$.

b) Prove that the assignment $X \mapsto X^t$ defines a diffeomorphism from $\mathrm{Sp}(2n)$ onto itself.

II.2 Let v be a smooth vector field on a manifold M and $x^1, \dots, x^n$ be coordinates in a neighborhood of a zero $p_0 \in M$ of v.

a) When do we say that the zero p_0 of v is *non-degenerate*?

b) Consider v as the smooth map $M \to TM$ and compute the matrix of the differential of this map at p_0 with respect to $x^1, \dots, x^n$ and the corresponding local coordinates near $(p_0, \vec{0}) \in TM$.

c) Prove that if the map $v : M \to TM$ intersects the zero section of TM transversely, then all zeros of v are non-degenerate.

II.3

a) Define the topological space $\mathbb{RP}^n$ and describe your favorite CW structure on this space.

b) Let X be the space obtained by gluing two copies of $\mathbb{RP}^2$ to one another along a loop that represents a generator for $\pi_1(\mathbb{RP}^2)$. Describe the universal cover of the space X.

c) Describe all finite connected coverings of the space X in **b)** up to equivalence.