Ph.D. Comprehensive Examination  
Differential Geometry and Topology  
August 2014

Part I. Do three of these problems.

I.1 Let \( Y \) be a torus, \( D_0 \) a closed disk in \( Y \), let \( D \) be the closed disk in \( \mathbb{R}^2 \), and let \( X \) be the result of gluing the boundary of \( D \) to the boundary of \( D_0 \).

a) Use van Kampen’s Theorem to find \( \pi_1(X) \).

b) Use the Mayer-Vietoris Theorem to find \( H_k(X) \).

I.2 Let \( X \) be a smooth manifold. Give a careful definition of its tangent bundle \( TX \). Then prove that \( T(\mathbb{RP}^2) \) is not diffeomorphic to \( \mathbb{RP}^2 \times \mathbb{R}^2 \).

I.3 Let \( X \) and \( Y \) be smooth connected manifolds (without boundary) of the same dimension with \( X \) compact. Suppose \( f : X \rightarrow Y \) is an immersion (that is, \( df_x : T_x X \rightarrow T_{f(x)} Y \) is injective for every \( x \)). Show that \( f \) is onto.

I.4 Prove that \( S^n \) admits a non-vanishing continuous vector field if and only if \( n \) is odd.

Part II. Do two of these problems.

II.1 Let \( X = S^2 \cup A \) be the union of a 2-sphere and an axis connecting the north and south poles.

a) Compute \( \pi_1(X) \).

b) Explain why \( X \) has a universal covering \( \tilde{X} \), describe it (including a local picture), and describe the action of \( \pi_1(X) \) on \( \tilde{X} \).

c) Why does \( X \) admit an \( n \)-sheeted covering for each \( n \in \mathbb{N} \)?

II.2 Let \( X \) and \( Y \) be smooth manifolds, \( f : X \rightarrow \mathbb{R} \) and \( g : X \rightarrow Y \) smooth with \( g \) a submersion (that is, \( dg_x : T_x X \rightarrow T_{g(x)} Y \) is onto for every \( x \)).

a) Let \( y_0 \in g(X) \) and let \( Z = g^{-1}(y_0) \). Show that \( Z \) is a smooth manifold.
b) Let $h : Z \to \mathbb{R}$ be the restriction of $f$ to $Z$. Show that $x_0$ is a critical point of $h$ (that is, $dh(x_0) = 0$) if and only if $df(x_0)$ belongs to the image of

$$dg_{x_0}^* : T_{g(x_0)}^* Y \to T_{x_0}^* X.$$ 

Remark: This is what is behind the method of Lagrange multipliers.

II.3 Let $X$ be a smooth closed (compact without boundary) orientable $n$-dimensional manifold, and pick an orientation. Let $\Omega^k(X) = C^\infty(X; \wedge^k X)$ be the space of smooth $k$-forms on $X$ and denote by $H^k_{dR}(X)$ the $k$-th de Rham cohomology group.

a) Explain why integration of smooth $n$-forms is well defined.

Define $\beta : \Omega^k(X) \times \Omega^{n-k}(X) \to \mathbb{R}$ by

$$\beta(\phi, \psi) = \int_X \phi \wedge \psi.$$ 

b) Let $\phi \in \Omega^k(X)$. Show that

$$\beta(\phi, d\chi) = 0 \text{ for all } \chi \in \Omega^{n-k-1}(X) \iff \phi \text{ is closed}.$$ 

c) Show that $\beta$ determines a map

$$H^k_{dR}(X) \times H^{n-k}_{dR}(X) \to \mathbb{R}.$$ 

Hint: Part c) only needs the easier direction of part b).