Ph.D. Comprehensive Examination
Complex Analysis

January 2021

Part I. Do three of these problems.

I.1. Compute using contour integration

$$\int_0^{2\pi} \frac{\cos\theta}{\cos\theta - i} d\theta.$$

I.2. Let $G = \{z \in \mathbb{C} : |z| < 1\}$. Let

$$K = \left\{ z \in \mathbb{C} : \frac{1}{4} \leq |z| \leq \frac{3}{4} \right\}.$$

Show that there exists a function f analytic on some open set G_1 containing K, which cannot be approximated by functions analytic in G.

I.3. Give an example of an unbounded function, analytic in $\mathbb{C} \setminus \{z \in \mathbb{C} : \mathrm{Im}\,(z) = 0,\ \mathrm{Re}\,(z) \leq 0\}$ (complex plane cut along the negative real line), such that

$$\limsup_{z \to x} |f(z)| \leq 1, \quad \forall x \leq 0.$$

I.4. Suppose that a sequence of analytic functions $f_n : \mathcal{H}_+ = \{z \in \mathbb{C} : \mathrm{Im}\,(z) > 0\} \to \mathbb{C}$ satisfies $\mathrm{Im}\,(f_n(z)) > 0$ for all $z \in \mathcal{H}_+$ and all $n \geq 1$. Suppose that for each $z \in \mathcal{H}_+$ the limit

$$f(z) = \lim_{n \to \infty} f_n(z)$$

exists. Prove that $f(z)$ is analytic in $\mathcal{H}_+$ and that convergence is uniform on compact subsets of $\mathcal{H}_+$.

Part II. Do two of these problems.

II.1. Let $G = \{z \in \mathbb{C} : |z| > 1\}$. Suppose $f : G \to \mathbb{C}$ is analytic and there exists a sequence $z_n \to \infty$, such that $z_n^2 f'(z_n) \to 0$. Prove that f cannot be injective on G.

II.2. Let f be analytic in $B^-(0, R) = \{z \in \mathbb{C} : |z| \leq R\}$ with $f(0) = 0$, $f'(0) \neq 0$, and $f(z) \neq 0$ for $0 < |z| \leq R$. Put

$$\rho = \min\{|f(z)| : |z| = R\} > 0.$$

Define $g : B(0, \rho) \to \mathbb{C}$ by

$$g(\omega) = \frac{1}{2\pi i} \int_{|z|=R} \frac{z f'(z)}{f(z) - \omega} dz.$$

Show that $z = g(\omega)$ is the unique solution of $f(z) = \omega$, $|z| < R$, provided $\omega \in B(0, \rho)$.
Hint: Use argument principle to show uniqueness of solution and the residue theorem to show that the solution is $g(\omega)$.

II.3. A Poincaré line is an arc of a circle orthogonal to the unit circle $|z| = 1$ that lies inside the unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. Use conformal automorphisms of the unit disk U to write parametric equations of the Poincaré line passing through two given points $\{a, b\} \subset U$.