Ph.D. Comprehensive Examination
Complex Analysis

January 2019

Justify your answers thoroughly. For any theorem that you wish to cite, you should either give its name or a statement of the theorem.

Part I. Do three of these problems.

I.1. Let $f(z) = \dfrac{z^3 \sin(1/z)}{(z+1)^2}$.

(a) Classify the singularities of f as removable, a pole (include the order of the pole), or an essential singularity.

(b) Evaluate $\displaystyle\int_{|z|=2} f(z)\,dz$, where the circle of integration has the counter-clockwise orientation.

I.2. (a) Show that an entire function $f(z)$ is a nonconstant polynomial if and only if $|f(z)| \to \infty$ as $|z| \to \infty$.

(b) Give an explicit example of an entire function $g(z)$ such that $|g(z)|$ has no limit as $|z| \to \infty$ (and justify your assertion).

I.3. Let $f(z)$ and $g(z)$ be polynomials with $\deg(g) > \deg(f) + 1$. Prove that the sum of the residues of $f(z)/g(z)$ at all its poles is zero.

I.4. Show that the function f defined by

$$f(z) = \frac{1}{1-z^2}, \quad z \in G = \mathbb{C} \setminus \{z \in \mathbb{R} : |z| \le 1\}$$

has a square root in G.

Part II on next page

Part II. Do two of these problems.

II.1. Let $p(z)$ be a nonconstant polynomial. Show that for any $c > 0$, every connected component of $\{z : |p(z)| < c\}$ has a zero of $p(z)$.

II.2. Let $f(z)$ be meromorphic on $\mathbb{C}$ with a pole only at the origin, let $r > 0$ and define

$$g(z) = \frac{1}{2\pi i} \int_{|\zeta|=r} \frac{f(\zeta)}{\zeta - z}\, d\zeta, \quad |z| > r,$$

where the the circle of integration is oriented counterclockwise. Show that g has an extension to all of $\mathbb{C}$ as a meromorphic function and that $f(z) + g(z)$ is entire.

II.3. Let $p_n(z) = \sum_{k=0}^{n} \frac{z^{2k}}{(2k)!}$. This is the $2n^{\text{th}}$-degree Taylor polynomial of $\cosh(z) = (e^z + e^{-z})/2$ centered at 0. Show that for any $r > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $p_n(z)$ has no zeros in $\Omega_r = \{z : |\operatorname{Re} z| < r,\ |\operatorname{Im} z| < \pi/3\}$.