Ph.D. Comprehensive Examination
Complex Analysis

January 2015

Part I. Do three of these problems.

I.1 Let $f : B(0,2) \longrightarrow \mathbb{C}$ be an analytic function satisfying $|f(z) - 2| < 1$ for each $z \in \mathbb{C}$ such that $|z| = 1$. Show that:

(a) $|f(z)| < 3$ for each $z \in B(0,1)$;

(b) $f(z) \neq 0$ for each $z \in B(0,1)$.

I.2 Let G be an open subset of $\mathbb{C}$ and consider $f : G \to \mathbb{C}$ an analytic function which is one-to-one. Show that $f'(z) \neq 0$ for each $z \in G$.

I.3 Let $\gamma : \left[0, \frac{\pi}{2}\right] \longrightarrow \mathbb{C}$ be given by $\gamma(t) := 2e^{it}$. Compute $\displaystyle\int_\gamma (z^2 - 3|z| + \operatorname{Im} z)\, dz$.

I.4 Suppose that u is a real-valued harmonic function in $B(0,1) \subseteq \mathbb{C}$, such that u^2 is also harmonic in $B(0,1)$. Prove that u is constant.

Justify your answers thoroughly. For any theorem that you wish to cite, you should either give its name or a statement of the theorem. For each $z_o \in \mathbb{C}$ and $r > 0$, the open ball in $\mathbb{C}$ with center z_o and radius r is denoted by $B(z_o, r)$. $\mathcal{H}(G)$ is the space of holomorphic functions $G \to \mathbb{C}$.

Part II. Do two of these problems.

II.1 Let $D = \{z \in \mathbb{C} : |z| < 1\}$ and $\{f_n\}_{n=1}^{\infty}$ a sequence of continuous functions on $\mathrm{cl}(D)$, the closure of D, which are holomorphic in D and such that $f_n(\partial D) \subset \partial D$. Suppose $f_n \to f$ uniformly in $\mathrm{cl}(D)$. Show that either $f(D) \subseteq D$ or $f(z) = e^{i\theta}$ for all $z \in D$ and some $\theta \in \mathbb{R}$.

II.2 Let $0 < r_0 < r_1$ and $0 < R_0 < R_1$. Let G be the annulus $\{z \in \mathbb{C} : r_0 < |z| < r_1\}$. Suppose $f \in \mathcal{H}(G) \cap C(\mathrm{cl}(G))$ has no zeros in G and satisfies $|f(z)| = R_i$ if $|z| = r_i$, $i = 1, 2$. Show that f maps G into the annulus $\{z \in \mathbb{C} : R_0 < |z| < R_1\}$.

II.3 Let $\varepsilon > 0$, $I = (-\varepsilon, \varepsilon) \subset \mathbb{R}$, and $\gamma : I \to \mathbb{C}$ a curve. Suppose γ is one-to-one and given by a power series $\gamma(t) = \sum_{n=0}^{\infty} a_n (t - t_0)^n$ which converges in all of I.

(a) Show that there a holomorphic function $\Gamma : B(0, \varepsilon) \to \mathbb{C}$ such that $\Gamma(t) = \gamma(t)$ when $t \in I \cap B(0, \varepsilon)$.

(b) Suppose $a_1 \neq 0$. Show that with suitable $0 < r < \varepsilon$, the restriction of Γ to $B(0, r)$ is bijective onto its image.

Assuming now $a_1 \neq 0$ and r as in the previous item, let $U = \Gamma(B(0, r))$. Then $U \backslash (U \cap \gamma(I))$ has two connected components which we label U_+ and U_-.

(c) Show that if f is continuous in $U_+ \cup (U \cap \gamma(I))$, holomorphic in U_+, and real-valued on $U \cap \gamma(I)$, then there is g holomorphic in U such that $g = f$ in U_+.