• Justify your answers thoroughly.
• Notation: $\mathbb{C}$ denotes the set of complex numbers. For each $z_0 \in \mathbb{C}$ and $r > 0$ the open ball in $\mathbb{C}$ with center $z_0$ and radius $r$ is denoted by $B(z_0, r)$. If $E \subseteq \mathbb{C}$ then $\overline{E}$ stands for the closure of the set $E$ in $\mathbb{C}$.
• For any theorem that you wish to cite, you should either give its name or a statement of the theorem.

Part I. (Do 3 problems):

I.1. Consider the curve $\gamma$ given by $\gamma(t) := 1 + e^{it}$, for $0 \leq t \leq 2\pi$. For each positive integer $n$ evaluate
\[
\int_{\gamma} \left( \frac{z}{z-1} \right)^n \, dz.
\]

I.2. Let $f : \mathbb{C} \setminus \{0, 1, 2\} \to \mathbb{C}$ be given by $f(z) := \frac{1}{z(z-1)(z-2)}$. Give the Laurent expansion of $f$ in each of the following annuli:
   (a) $A = \{z \in \mathbb{C} : 0 < |z| < 1\}$
   (b) $B = \{z \in \mathbb{C} : 1 < |z| < 2\}$
   (c) $C = \{z \in \mathbb{C} : 2 < |z|\}$.

I.3. Let $f : B(0, 2) \to \mathbb{C}$ be an analytic function. Show that
\[
\max_{|z|=1} \left| \frac{1}{z} - f(z) \right| \geq 1.
\]

I.4. Suppose that the function $u : \mathbb{R}^2 \to \mathbb{R}$ is harmonic and that there exist $a, b \in \mathbb{R}$, $a < b$ with $u(x, y) \not\in (a, b)$ for each $(x, y) \in \mathbb{R}^2$. Show that $u$ is constant.
Part II. (Do 2 problems):

II.1. Let $\Omega := \{z \in \mathbb{C} : |\text{Re } z| < 1 \text{ and } |\text{Im } z| < 1\}$ and consider the function $f : \overline{\Omega} \to \mathbb{C}$ continuous on $\overline{\Omega}$, analytic in $\Omega$, and with the property that $f(z) = 0$ when $\text{Re } z = 1$. Prove that $f$ is identically zero in $\Omega$.

II.2. Let $G$ be an open and connected set in the complex plane and suppose that $\{f_n\}_{n \in \mathbb{N}}$ is a sequence of analytic functions defined on $G$ which converges uniformly on $G$ to a function $f : G \to \mathbb{C}$. Show that $f$ is an analytic function.

II.3. Let $r > 0$ and consider an analytic function $f : B(0, r) \to \mathbb{C}$ such that $f(0) = 0$ and there exists $A \in \mathbb{R}, A > 0$, with the property that $\text{Re } f(z) < A$ for each $z \in B(0, r)$. Show that 

$$|f(z)| \leq \frac{2A|z|}{r - |z|} \quad \forall z \in B(0, r).$$