Complex Analysis Ph.D. Qualifying Exam

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- Justify your answers thoroughly.
- Notation: $\mathbb{C}$ denotes the set of complex numbers. For each $z_o \in \mathbb{C}$ and $r > 0$ the open ball in $\mathbb{C}$ with center z_o and radius r is denoted by $B(z_o, r)$. If $E \subseteq \mathbb{C}$ then $\overline{E}$ stands for the closure of the set E in $\mathbb{C}$.
- For any theorem that you wish to cite, you should either give its name or a statement of the theorem.

Part I. (Do 3 problems):

I.1. Consider the curve γ given by $\gamma(t) := 1 + e^{it}$, for $0 \leq t \leq 2\pi$. For each positive integer n evaluate

$$\int_\gamma \left(\frac{z}{z-1}\right)^n dz.$$

I.2. Let $f : \mathbb{C} \setminus \{0,1,2\} \longrightarrow \mathbb{C}$ be given by $f(z) := \dfrac{1}{z(z-1)(z-2)}$. Give the Laurent expansion of f in each of the following annuli:
(a) $A = \big\{z \in \mathbb{C} :\, 0 < |z| < 1\big\}$
(b) $B = \big\{z \in \mathbb{C} :\, 1 < |z| < 2\big\}$
(c) $C = \big\{z \in \mathbb{C} :\, 2 < |z|\big\}$.

I.3. Let $f : B(0,2) \longrightarrow \mathbb{C}$ be an analytic function. Show that

$$\max_{|z|=1} \left| \frac{1}{z} - f(z) \right| \geq 1.$$

I.4. Suppose that the function $u : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is harmonic and that there exist $a, b \in \mathbb{R}$, $a < b$ with $u(x,y) \notin (a,b)$ for each $(x,y) \in \mathbb{R}^2$. Show that u is constant.

Part II. (Do 2 problems):

II.1. Let $\Omega := \{z \in \mathbb{C} : \ |\text{Re}\, z| < 1 \ \text{ and } \ |\text{Im}\, z| < 1\}$ and consider the function $f : \overline{\Omega} \to \mathbb{C}$ continuous on $\overline{\Omega}$, analytic in Ω, and with the property that $f(z) = 0$ when $\text{Re}\, z = 1$. Prove that f is identically zero in Ω.

II.2. Let G be an open and connected set in the complex plane and suppose that $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of analytic functions defined on G which converges uniformly on G to a function $f : G \to \mathbb{C}$. Show that f is an analytic function.

II.3. Let $r > 0$ and consider an analytic function $f : B(0, r) \longrightarrow \mathbb{C}$ such that $f(0) = 0$ and there exists $A \in \mathbb{R}$, $A > 0$, with the property that $\text{Re}\, f(z) < A$ for each $z \in B(0, r)$. Show that

$$|f(z)| \leq \frac{2A|z|}{r - |z|} \qquad \forall\, z \in B(0, r).$$