Ph.D. Comprehensive Examination
Complex Analysis

January 2013

Part I. Do three of these problems.

I.1 Compute $\displaystyle\int_0^\infty \frac{\log x}{1+x^4}\,dx$. Prove all claims.

I.2 Let $D = \{z \in \mathbb{C} : |z| < 1\}$. Suppose $f : \overline{D} \to \mathbb{C}$ is continuous, holomorphic in D, and $f(\partial D) \subset \mathbb{R}$. Define $h : \mathbb{C} \to \mathbb{C}$ by

$$h(z) = \begin{cases} f(z) & \text{if } z \in \overline{D} \\ \overline{f(1/\overline{z})} & \text{if } z \in \mathbb{C}\backslash\overline{D}. \end{cases}$$

Show that

(a) h is holomorphic in $\mathbb{C}\backslash\overline{D}$.

(b) h is entire. Hint: Use Morera's theorem.

(c) Show that $f(z)$ is a constant function.

I.3 Let $D = \{z \in C : |z| < 1\}$ and

$$E = \left\{w \in \mathbb{C} : \left(\frac{\operatorname{Re} w}{a}\right)^2 + \left(\frac{\operatorname{Im} w}{b}\right)^2 < 1\right\}$$

where a and b are positive numbers. Let $f : E \to D$ be holomorphic. Show that

$$|f'(0)| \le \max\left\{\frac{a}{b^2}, \frac{b}{a^2}\right\}.$$

Hints: If $0 < r < 1$ is close to 1 then $\gamma(t) = ra\cos t + irb\sin t$ lies in E and close to ∂E. You may use without proof the fact that the critical points of

$$\rho(t) = \frac{(a^2\sin^2 t + b^2\cos^2 t)^{1/2}}{a^2\cos^2 t + b^2\sin^2 t}$$

are exactly the zeros of $\sin 2t$.

I.4 Let $f(z) = \displaystyle\sum_{k=0}^{\infty} a_k z^k$ be entire and not a polynomial. Let $p_n(z) = \displaystyle\sum_{k=0}^{n} a_k z^k$ be its n-th Taylor polynomial centered at 0, and let $r_n = \sup\{|z| : p_n(z) = 0\}$. Show that $\displaystyle\lim_{n\to\infty} r_n = \infty$.

Part II. Do two of these problems.

II.1 Let $f(z)$ be analytic in an open set $G \subset \mathbb{C}$ except for a pole at $z_0 \in G$.

(a) Show that for any $\delta > 0$ there exists an $R > 0$ such that

$$\{w \in \mathbb{C} : \ |w| > R\} \subset f(\{z \in \mathbb{C} : \ |z - z_0| < \delta\}).$$

(b) Let $g(z)$ be an entire function that is not a polynomial. Show that $g(f(z))$ has an essential singularity at $z = z_0$.

II.2 Let $\omega : \mathbb{R} \to \mathbb{R}$ be of class C^∞ with $\omega(0) = 1$ and $\omega(t) = 0$ if $t > 1$. Define $f(z) = \displaystyle\int_0^\infty t^{z-1}\omega(t)\, dt$.

(a) Show that $f(z)$ is holomorphic in $\operatorname{Re} z > 0$.

(b) Show that $f(z)$ has a meromorphic extension to all of $\mathbb{C}$ with exception of simple poles at $z = 0, -1, -2, \ldots$ Hint: use integration by parts.

(c) Find $\operatorname{Res}(f; 0)$.

II.3 Let $G \subset \mathbb{C}$ be a region and let $D = \{z \in \mathbb{C} : |z| < 1\}$. Let G^c denote the complement of G in $\mathbb{C}$. Suppose there exists $a \in G^c$ and a $\delta > 0$ such that $\{z \in \mathbb{C} : \ |z - a| < \delta\} \subset G^c$. Show that

(a) There exists an injective function $h : G \to D$.

(b) Let $z_0 \in G$, let $q = \sup\{|f'(z_0)| : f \in \operatorname{Hol}(G, D)\}$. Show that there exists $f \in \operatorname{Hol}(G, D)$ such that $|f'(z_0)| = q$.

(c) Suppose $f \in \operatorname{Hol}(G, D)$ is such that $|f'(z_0)| = q$. Show that $f(z_0) = 0$.