Ph.D. Comprehensive Examination in Complex Analysis
Department of Mathematics, Temple University

January, 2011

Part I: Do three of the following problems

1. Suppose $f(z)$ is analytic in $\mathbb{C} - \{0\}$ and satisfies the inequality

$$|f(z)| \leq |z| + \frac{1}{|z|^{3/2}}.$$

Prove that $zf(z)$ is a polynomial of degree less or equal to 2.

2. Use the theory of residues to evaluate $\displaystyle\int_0^\infty \frac{\cos(ax)}{(x^2+1)^2}\,dx$, where $a > 0$ is a real constant.

3. Let $f(z) = \displaystyle\int_0^\infty \frac{\tan^2(zt)}{t^2}\,dt$. Prove that $f(z)$ is analytic in the upper half-plane $H = \{z \in \mathbb{C} : \Im(z) > 0\}$ and find an expression for $f'(z)$, $z \in H$.

4. Let $f(z)$ be an analytic function from the open unit disc D to itself. Suppose $f(0) = f(\frac{i}{2}) = 0$. Prove that $|f'(0)| \leq \frac{1}{2}$ and $|f(\frac{-i}{2})| \leq \frac{2}{5}$.

Part II: Do two of the following problems

1. Let $f(z)$ be an entire function that is not a polynomial.

(a) For $R > 0$, let $U_R = \{z \in \mathbb{C} : |z| > R\}$. Prove that $f(U_R)$ is an open everywhere dense subset of $\mathbb{C}$.

(b) Use the result of Part (a) to prove that there exists an everywhere dense subset $\Omega \subseteq \mathbb{C}$ such that for every $w \in \Omega$ there exist infinitely many $z \in \mathbb{C}$ with $f(z) = w$.

2. Let $f(z)$ be analytic on an open set containing 0. Suppose $f(z)$ has a zero of order n at $z = 0$. Show that there exist a $\delta > 0$ and an $\epsilon > 0$ such that for any $w \in B(0; \epsilon) - \{0\}$, $f(z) = w$ has n distinct solutions in $B(0; \delta)$.

3. Let $G_1 = \{z \in \mathbb{C} : 0 < |z| < 1\}$ and $G_2 = \{z \in \mathbb{C} : r < |z| < R\}$ where $R > r > 0$.

(a) Show that there exists a conformal mapping from G_1 onto G_2. Hint: find a conformal map from a vertical strip onto G_2.

(b) Show that there exists no bijective conformal mapping from G_1 onto G_2.