Part I: Do three of the following problems

1. Suppose $f(z)$ is analytic in $\mathbb{C} - \{0\}$ and satisfies the inequality
   \[ |f(z)| \leq |z| + \frac{1}{|z|^{3/2}}. \]

   Prove that $zf(z)$ is a polynomial of degree less or equal to 2.

2. Use the theory of residues to evaluate $\int_0^\infty \frac{\cos(ax)}{(x^2 + 1)^2} \, dx$, where $a > 0$ is a real constant.

3. Let $f(z) = \int_0^\infty \frac{\tan^2(zt)}{t^2} \, dt$. Prove that $f(z)$ is analytic in the upper half-plane $H = \{z \in \mathbb{C} : \Im(z) > 0\}$ and find an expression for $f'(z)$, $z \in H$.

4. Let $f(z)$ be an analytic function from the open unit disc $D$ to itself. Suppose $f(0) = f\left(\frac{i}{2}\right) = 0$. Prove that $|f'(0)| \leq \frac{1}{2}$ and $|f\left(\frac{-1}{2}\right)| \leq \frac{2}{5}$. 
Part II: Do two of the following problems

1. Let \( f(z) \) be an entire function that is not a polynomial.
   
   (a) For \( R > 0 \), let \( U_R = \{ z \in \mathbb{C} : |z| > R \} \). Prove that \( f(U_R) \) is an open everywhere dense subset of \( \mathbb{C} \).
   
   (b) Use the result of Part (a) to prove that there exists an everywhere dense subset \( \Omega \subseteq \mathbb{C} \) such that for every \( w \in \Omega \) there exist infinitely many \( z \in \mathbb{C} \) with \( f(z) = w \).

2. Let \( f(z) \) be analytic on an open set containing 0. Suppose \( f(z) \) has a zero of order \( n \) at \( z = 0 \). Show that there exist a \( \delta > 0 \) and an \( \epsilon > 0 \) such that for any \( w \in B(0; \epsilon) - \{0\} \), \( f(z) = w \) has \( n \) distinct solutions in \( B(0; \delta) \).

3. Let \( G_1 = \{ z \in \mathbb{C} : 0 < |z| < 1 \} \) and \( G_2 = \{ z \in \mathbb{C} : r < |z| < R \} \) where \( R > r > 0 \).
   
   (a) Show that there exists a conformal mapping from \( G_1 \) onto \( G_2 \). Hint: find a conformal map from a vertical strip onto \( G_2 \).
   
   (b) Show that there exists no bijective conformal mapping from \( G_1 \) onto \( G_2 \).