Ph.D. Comprehensive Examination
Complex Analysis Section
January 2010

Part I. Do three of these problems.

I.1. Evaluate the improper integral

$$\int_0^\infty \frac{x \sin x}{1+x^2}\, dx.$$

Prove all claims.

I.2. Let $f(z)$ and $g(z)$ be entire. Suppose that for any $z \in \mathbb{C}$, $|f(z)| \le |g(z)|$. Prove that $f(z) = cg(z)$ for some $c \in \mathbb{C}$ with $|c| \le 1$.

I.3. Let

$$s_n(z) = \sum_{k=-n}^{n} \frac{1}{z-k}, \quad n = 1, 2, \ldots$$

and let $s(z) = \lim_{n\to\infty} s_n(z)$. Show that the sequence $\{s_n(z)\}$converges uniformly on every open bounded set $G \subset \mathbb{C}$ with $\overline{G} \cap \mathbb{Z} = \emptyset$ and that $s(z)$ is a meromorphic function of z with simple poles at $z = k$, $k \in \mathbb{Z}$. Moreover, show that $s(z)$ is periodic with period 1.

I.4. Let $D = \{z : |z| < 1\}$. Find $u : D \to \mathbb{R}$ harmonic and such that for any $z_0 \in \partial D$ with $\operatorname{Re} z_0 \neq 0$,

$$\lim_{z\to z_0} u(z) = \begin{cases} 1 & \text{if } \operatorname{Re} z_0 > 0 \\ -1 & \text{if } \operatorname{Re} z_0 < 0. \end{cases}$$

Hint: Can you solve a similar problem on the strip $\{\zeta : -1 < \operatorname{Im} \zeta < 1\}$?

Part II. Do two of these problems.

II.1. Let $G \subset \mathbb{C}$ be a bounded open set, $\{f_k\}_{k=1}^\infty$ a sequence of continuous functions $\overline{G} \to \mathbb{C}$, holomorphic in G. Suppose that the sequence converges uniformly on the boundary of G to some function. Show that $\{f_k\}_{k=1}^\infty$ converges in G to an analytic function.

II.2. Let $G \subset \mathbb{C}$ be open, $a \in G$, and $r > 0$ such that $\overline{B(a,r)} \subset G$. Let $f : [0,1] \times G \to \mathbb{C}$ be continuous, holomorphic in $z \in G$, and such that $f(t,z) \neq 0$ when $t \in [0,1]$ and $|z-a| = r$. Show that the functions $z \mapsto f(0,z)$ and $z \mapsto f(1,z)$ have the same number of zeros in $B(a,r)$ counting multiplicity.

II.3. Let G be a simply connected domain, $G \neq \mathbb{C}$, $a \in G$, and $f : G \to D$ a bijective analytic map from G to the open unit disc $D = \{z \in \mathbb{C} : \ |z| < 1\}$ such that $f(a) = 0$. Show that for any analytic map $g : G \to D$ such that $g(a) = 0$, $|g'(a)| \le |f'(a)|$. Moreover, show that if $|g'(a)| = |f'(a)|$, then $g(z)$ is also a bijective analytic map from G to D.