Ph.D. Comprehensive Examination in Complex Analysis
Department of Mathematics, Temple University

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Part I: Do three of the following problems

1. Show that the function $u(x, y) = \log(x^2 + y^2)$ is harmonic in $\mathbb{C} \setminus \{0\}$ but that it has no harmonic conjugate in $\mathbb{C} \setminus \{0\}$.

2. Let $f(z)$ be an entire function that satisfies $\int_0^{2\pi} |f(re^{i\theta})|\, d\theta \leq r^{17/3}$ for all $r \geq 0$. Prove that $f(z) \equiv 0$.

3. Let $f(t)$ be a continuous real-valued function on the interval $[0, 1]$. Set $h(z) = \int_0^1 f(t) \cos(zt)\, dt$.

(a) Prove that $h(z)$ is an entire function.

(b) Prove that if $h(z) \equiv 0$, then $f(t) \equiv 0$.

4. Evaluate $\int_0^\infty \frac{x - \sin x}{x^3}\, dx$.

Part II: Do two of the following problems

1. Let U and V be two open connected subsets of $\mathbb{C}$ and let $f : U \to V$ be an analytic function on U. Suppose that for any $K \subset V$ compact $f^{-1}(K)$ is compact. Show that $f(U) = V$.

2. Suppose $f(z)$ is analytic on the right half-plane $H = \{z : \ \mathrm{Re}(z) > 0\}$ and satisfies $|f(z)| \leq 1$ for all $z \in H$, $f(1) = 0$. Find the largest possible value of $|f'(1)|$ and determine all functions $f(z)$ for which $|f'(1)|$ is the largest possible.

3. Let $a \in \mathbb{C}$ and let $\epsilon > 0$. Show that $f(z) = \sin z + \frac{1}{z - a}$ has infinitely many zeros in the strip $|\mathrm{Im}(z)| < \epsilon$.