PH.D. COMPREHENSIVE EXAMINATION
COMPLEX ANALYSIS SECTION

January, 2006

Part I. Do three (3) of these problems.

I.1. Find a conformal one-to-one map of the half disc \( \{ z : |z| < 2, \text{Re} z < 0 \} \) onto the unit disc \( U = \{ z : |z| < 1 \} \).

I.2. Let \( f(z) = u(z) + iv(z) \) be analytic in the unit disc \( U = \{ z : |z| < 1 \} \), \( u \) and \( v \) real. Show that if \( u(0)^2 = v(0)^2 \), then

\[
\int_0^{2\pi} u(re^{i\theta})^2 \, d\theta = \int_0^{2\pi} v(re^{i\theta})^2 \, d\theta \quad \text{for} \quad 0 < r < 1.
\]

I.3. Let \( f : [a, b] \rightarrow \mathbb{C} \) be continuous \((a < b)\). Let \( g \) be defined by

\[
g(z) = \int_a^b \frac{f(t)}{z-t} \, dt.
\]

Prove that \( g \) is analytic in \( \mathbb{C} \setminus [a, b] \).

I.4. Suppose \( f \) is meromorphic on \( \mathbb{C} \) and there exist \( K, k, R > 0 \) such that \( |f(z)| < K|z|^k \) if \( |z| > R \). Prove that \( f \) is a rational function.
Part II. Do two (2) of these problems.

II.1. 
(a) Let $U$ be the unit disc. Suppose $f(z)$ is analytic in $U$, continuous on $\overline{U}$ and real-valued on $\partial U$. Prove that $f$ is constant.
(b) If $f(z)$ is as in (a), except that $f(z)$ is assumed real-valued only on the arc $\gamma = \{e^{i\theta} : 0 < \theta < \frac{\pi}{5}\}$, what can we conclude? Explain.

II.2. Let $f$ be analytic in the region $|z| > 1$. Prove that if $f$ is real-valued on $(1, \infty)$, then it is also real-valued on $(-\infty, -1)$.

II.3. Let $\{f_n\}$ be a sequence of analytic functions on a region $G$ that converges uniformly on $G$. If $K$ is a compact subset of $G$, prove that the sequence of derivatives $\{f'_n\}$ converges uniformly on $K$. 