January 12, 2005

Comprehensive Examination Department of Mathematics

Complex Analysis

Part I: Do three of the following problems

1. (a) Find the radius of convergence R of the power series $\sum_{n=0}^{\infty} z^{n!}$.

(b) Let $f(z) = \sum_{n=0}^{\infty} z^{n!}$, $|z| < R$. Show that $f(z)$ cannot be continued analytically to any region $G \underset{\neq}{\supset} D$ where $D = \{z \in \mathbf{C} : \ |z| < R\}$.

2. Suppose $f(z)$ is analytic in the entire complex plane with an exception of a pole of order 2 at $z = 0$ and a simple pole at $z = 1$. Suppose further that $|f(z)| \leq K|z|^4$ for all z with $|z| \geq 2$, where $K > 0$ is a constant.

(a) Show that $f(z) = \frac{P(z)}{Q(z)}$ where $P(z)$ and $Q(z)$ are polynomials

(b) Suppose further that $P(z)$ and $Q(z)$ are relatively prime. What can you say about the degrees of $P(z)$ and $Q(z)$?

3. Suppose $f(z)$ is analytic in an open disc D and continuous on the boundary of D. Suppose further that $f(z)$ is purely imaginary on the boundary of D. Prove that $f(z)$ is a constant function.

4. (a) Identify the singularities of

$$f(z) = \frac{z^3 \sin(\frac{\pi}{z})}{(z-1)^2} \text{ in } \mathbf{C}.$$

Classify each singularity as a removable singularity, a pole (please include the order of the pole), or an essential singularity.

(b) Find $\oint_{|z|=2} f(z)dz$.

Part II: Do two of the following problems

1. Prove that if $f(z)$ is entire and one-to-one on $\mathbf{C}$, then $f(z) = az + b$ with $a \neq 0$.

2. Let G be a region that contains the closed unit disc $\bar{D}$, and let $\{f_n(z)\}$ be a sequence of analytic functions that converge uniformly to a function $f(z)$ on G. Suppose $f_n(z)$ have no zeros in $\bar{D}$. Show that either $f(z) \equiv 0$ in G or $f(z) \neq 0$ for any $z \in \bar{D}$.

3. (a) Show that

$$\sin z = z e^{g(z)} \prod_{n=1}^{\infty} (1 - \frac{z^2}{\pi^2 n^2}) \text{ where } g(z) \text{ is an entire function.}$$

(b) Given that $g(z) = 0$, i.e., given that $\sin z = z \prod_{n=1}^{\infty} (1 - \frac{z^2}{\pi^2 n^2})$, find a product representation for $e^z - 1$.