PH.D. COMPREHENSIVE EXAMINATION
COMPLEX ANALYSIS SECTION

January 1996

Part I. Do three (3) of these problems.

I.1. Let \( \mathbb{R}^- = \{ x \in \mathbb{R} : x \leq 0 \} \). Suppose \( f(z) \) is analytic on \( \mathbb{C} \setminus \mathbb{R}^- \) and \( f(x) = x^x \) for \( x \in \mathbb{R}, x > 0 \). Find \( f(i) \) and \( f(-i) \).

I.2. Let \( f(z) \) be an analytic function on an open connected subset \( G \subset \mathbb{C} \). Suppose that \( f(z) \) maps \( G \) onto a subset of a straight line. Show that \( f(z) \) is a constant.

I.3. Find a conformal mapping from the region \( \{ z \in \mathbb{C} : |z-1| > 1 \text{ and } |z+1| > 1 \} \) onto the punctured disc \( D = \{ z \in \mathbb{C} : |z| < 1 \}\setminus0 \). Hint: Apply \( T(z) = \frac{1}{z} \) first.

I.4. Evaluate \( \int_{-\infty}^{\infty} \frac{\cos x}{x^2+1} \, dx \) using residues.

Part II. Do two (2) of these problems.

II.1. Let \( G_1 \) and \( G_2 \) be two bounded simply connected regions, and let \( z_0 \in G_1 \) and \( w_0 \in G_2 \). Show that there exists a bijective analytic mapping \( f(z) \) from \( G_1 \) to \( G_2 \) such that \( f(z_0) = w_0 \).

II.2. Let \( \Gamma(z) = \int_0^\infty t^{z-1}e^{-t} \, dt, \text{ Re}(z) > 0. \) Show that \( \Gamma(z+1) = z\Gamma(z) \), use this formula to obtain a meromorphic continuation of \( \Gamma(z) \) to the entire complex plane, and find the poles of \( \Gamma(z) \) on \( \mathbb{C} \), their orders and residues.

II.3. i) Let \( u(x, y) \) be a harmonic function on the disc \( D = \{ z : |z-z_0| < R \} \). Show that for any \( r < R, u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta}) \, d\theta \).

ii) Let \( u(x, y) \) be a harmonic function on a bounded region \( G \) that is continuous on the closure \( \overline{G} \) of \( G \). Show that \( u(x, y) \) achieves its maximum and minimum values on the boundary of \( G \).