# Ph.D. Comprehensive Examination
# Complex Analysis
## August 2022

Justify your answers thoroughly. For any theorem that you wish to cite, you should either give its name or a statement of the theorem.

### Part I. Do three of these problems.

**I.1.** Consider the integral

$$I = \int_{-\infty}^{\infty} \frac{dx}{1 + (x - 2i)^2}.$$

Let us make the change of variable $z = x - 2i$. Then

$$I = \int_{-\infty}^{\infty} \frac{dz}{1 + z^2} = \pi.$$

Explain why the solution is wrong and compute the correct answer.

**I.2.** Let $D = \{z \in \mathbb{C} : |z| < 1\}$ and $f : D\backslash\{0\} \to \mathbb{C}$ holomorphic (analytic). Suppose that for some $\varepsilon > 0$, $|f(z)| \leq \big|\log |z|\big|$ if $0 < |z| < \varepsilon$. Show that the singularity at $z = 0$ is removable.

**I.3.** Suppose that $f$ is analytic in the punctured unit disc $D\backslash\{0\}$ and that $z = 0$ is an essential singularity for $f$. Prove that $f$ cannot be one-to-one on $D\backslash\{0\}$.

**I.4.** Give an example of a function $f$ analytic in the unit disc $D$ such that $f'(z) \neq 0$ for any $z \in D$ but $f$ is not injective on $D$. (Prove your assertions.)

Part II on next page

**Part II. Do two of these problems.**

**II.1.** Let

$$f(z) = \sum_{n=0}^{\infty} z^{n!}.$$

Prove that $f$ is analytic in the unit disc $D$ and cannot be analytically continued to any domain $G$ containing $D$.

**II.2.** Suppose $f$ is analytic in the unit disk and $|f(z)| < 1$ when $|z| < 1$. Suppose that $f(1/2) = 0$. Prove that $|f(0)| \leq 1/2$.

**II.3.** Let $G = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0,\ |z - 2i| > 1\}$. Find a conformal isomorphism between $G$ and the annulus $A = \{z \in \mathbb{C} : 1 < |z| < R\}$, where the value of $R$ is to be determined.