Ph.D. Comprehensive Examination
Complex Analysis

August 2019

Justify your answers thoroughly. For any theorem that you wish to cite, you should either give its name or a statement of the theorem.

Part I. Do three of these problems.

I.1. Prove or disprove the following statement: Let f be analytic on the unit ball B centered at the origin. Suppose $\{a_k\}$ is a sequence in B that converges to 1 and $f(a_k) = 0$ for every k. Then $f(z) = 0$ for all $z \in B$.

I.2. Is there a sequence $\{f_n\}_{n=1}^{\infty}$ of nowhere zero entire functions that converges to $f(z) = z$ uniformly on compact sets? (Remember: fully justify your answer.)

I.3. Suppose $\Omega \subset \mathbb{C}$ is open and $K \subset \Omega$ is closed with the property that there is $r > 0$ such that for every $a \in K$ and $f \in \mathcal{H}(\Omega)$ there is C such that

$$|f^{(n)}(a)| \leq C\frac{n!}{r^n} \quad \text{for all } n.$$

Show that

$$\{a + z : a \in K,\ |z| < r\} \subset \Omega.$$

I.4. Evaluate

$$\int_{-\infty}^{\infty} e^{-x^2+ix\xi}\,dx.$$

You may use the fact that

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}.$$

Part II. Do two of these problems.

II.1. Let $\Omega \subset \mathbb{C}$ be open. Suppose h is a complex valued harmonic function on Ω with the property that

$$\int_{B(z_0,r)} h(z) z^m\,d\lambda(z) = 0 \quad \text{for } m = 1, 2, \ldots$$

whenever $B(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\}$ has closure contained in Ω. Show that h is holomorphic. In the integral above, $d\lambda$ means Lebesgue measure.
Hint: If h is real-valued and harmonic on a disc, then there is a holomorphic function f on that disc such that $h(z) = f(z) + \overline{f(z)}$.

II.2. Let f be analytic on the punctured disc $B\backslash\{0\}$, where B is the unit ball centered at the origin. Assume that $\int_B |f(z)|\,d\lambda(z) < \infty$.

(a) Show that the singularity at the origin is either removable or a pole.

(b) Show that the singularity at the origin is either removable or a pole of order one.

Hint: For part (b), you can assume the following fact - Since f is integrable on B, for each $\epsilon > 0$, there is $r > 0$ such that $\int_{B_r} |f(z)|\, d\lambda(z) < \epsilon$ where B_r is the ball of radius r centered at the origin.

II.3. Let $z_1, z_2, z_3 \in \mathbb{C}$ be three non-collinear points. They are the vertices of a triangle whose sides (closed segments) we label L_1, L_2, L_3 with L_i opposite to z_i. Suppose $f(z)$ is meromorphic on $\mathbb{C}$ with poles only at these vertices, with respective residues r_1, r_2, r_3 satisfying $r_1 + r_2 + r_3 = 0$.

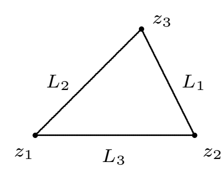
(a) Show that f has a primitive $g_i(z)$ in the region

$$G_i = \mathbb{C} \setminus \bigcup_{j \neq i} L_j.$$

Hint: The primitive may be defined using integration along suitable paths in G_i, but then argue why such integrals depend only on the endpoints of the path.

(b) Choose the primitives so that they all vanish at a given interior point a of the triangle. Assuming the labeling is as in the picture, show that

$$g_3(z) - g_2(z) + 2\pi i r_1 = 0$$



outside of the triangle.