Complex Analysis Ph.D. Qualifying Exam

Temple University

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- Justify your answers thoroughly.
- Notation: $\mathbb{C}$ denotes the set of complex numbers, $\mathbb{Z}$ denotes the set of integers and $\mathbb{N}$ denotes the set of natural numbers.
- For any theorem that you wish to cite, you should either give its name or a statement of the theorem.

Part I. (Do 3 problems):

I.1. Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be an entire function. Show that for each $z_o \in \mathbb{C}$ there holds:

$$\lim_{n\to\infty} \sqrt[n]{\frac{|f^{(n)}(z_o)|}{n!}} = 0.$$

Here, for each $n \in \mathbb{N}$, we let $f^{(n)}$ denote the derivative of order n of the function f.

I.2. Prove that the equation $e^z - z = 0$ has an infinite number of solutions $z \in \mathbb{C}$.

I.3. Identify all zeros and poles of the function $f : \mathbb{C} \setminus \{k\pi : k \in \mathbb{Z}\} \longrightarrow \mathbb{C}$ given by $f(z) := \dfrac{z}{\sin z}$ and determine their multiplicities (for the zeroes) and their orders (for the poles).

I.4. Let U, V be open sets in $\mathbb{C}$ and assume that $f : U \longrightarrow V$ is holomorphic and that $u : V \longrightarrow \mathbb{R}$ is harmonic. Prove that the function $\widetilde{u} : U \longrightarrow \mathbb{R}$ defined by $\widetilde{u} := u \circ f$ is harmonic on U.

Part II. (Do 2 problems):

II.1. Show that there exists no sequence of entire functions $\{f_n\}_{n\in\mathbb{N}}$ that converges uniformly to the function $f(z) := \dfrac{1}{z^2}$ on the unit circle $\big\{z \in \mathbb{C} : |z| = 1\big\}$ as n goes to infinity.

II.2. Let $u : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a harmonic function and suppose that

$$\iint_B u(x,y)\,dxdy \leq 2$$

for every ball B of radius 1 in $\mathbb{R}^2$. Show that u is constant.

II.3. Let f be an entire function with the property that, for each $w \in \mathbb{C}$, the equation $f(z) = w$ has exactly three solutions counted with multiplicity. Prove that f is a polynomial of degree three.