Complex Analysis Ph.D. Qualifying Exam

Temple University

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- Justify your answers thoroughly.
- Notation: $\mathbb{C}$ denotes the set of complex numbers.
- For any theorem that you wish to cite, you should either give its name or a statement of the theorem.

Part I. (Do 3 problems):

I.1. Let D be a region in $\mathbb{C}$ and $f : D \longrightarrow \mathbb{C}$ be a one-to-one holomorphic function. Show that for each $z \in D$ there holds $f'(z) \neq 0$.

I.2. Show that the function $f : \mathbb{C} \setminus \{0, 2\} \longrightarrow \mathbb{C}$ given by

$$f(z) := \frac{1}{z} + \frac{1}{z-2}, \quad \forall\, z \in \mathbb{C} \setminus \{0, 2\},$$

has no antiderivative on $\{z : 0 < |z| < 1\}$.

I.3. Let f be an entire function such that $\lim_{|z|\to\infty} |f(z)| = \infty$. Prove that f is a polynomial.

I.4. Give an example of a region $G \subset \mathbb{C}$ and a harmonic function $u : G \longrightarrow \mathbb{R}$ such that u is not the real part of any holomorphic function on G. You will need to justify your assertion.

Part II. (Do 2 problems):

II.1. Let $a \in \mathbb{C}$ and assume that $f : \mathbb{C} \longrightarrow \mathbb{C}$ is a function which is not identically zero, which is complex-differentiable at the origin and $f'(0) = a$, and which satisfies

$$f(z_1 + z_2) = f(z_1)f(z_2) \quad \text{for each} \quad z_1, z_2 \in \mathbb{C}.$$

Prove that $f(z) = e^{az}$ for each $z \in \mathbb{C}$.
(Hint: First show that f is an entire function).

II.2. Let $G \subset \mathbb{C}$ be a region and $\{f_n : n = 1, 2, 3, \dots\}$ be a sequence of holomorphic functions on G with the property that each f_n is zero-free on G. Assume that the sequence f_n converges uniformly to some f on G as $n \to \infty$. If f has a zero in G, show that f vanishes identically on G.

II.3. Let $D \subset \mathbb{C}$ be the unit disc and consider

$$\mathcal{F} = \{f : D \longrightarrow D,\ f \text{ holomorphic}\} \quad \text{and} \quad M := \sup_{f \in \mathcal{F}} |f'(0)|.$$

(a) Show that M is finite.
(b) Show that there exists $f \in \mathcal{F}$ satisfying $f'(0) = M$.