Ph.D. Comprehensive Examination
Complex Analysis
August 2015

Part I. Do three of these problems.

I.1 Let $D \subset \mathbb{C}$ be an open nonempty connected set and consider $h : D \rightarrow \mathbb{C}$ a holomorphic function satisfying

$$(h(z))^2 = \overline{h(z)} \quad \forall z \in D.$$ 

(a) Show that the function $h$ is constant on $D$.

(b) Find all possible values for the function $h$ satisfying the above property.

I.2 Consider the function $u : \mathbb{R}^2 \setminus \{(0,0)\} \rightarrow \mathbb{R}$ given by $u(x,y) := \ln(x^2 + y^2)$ for each $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$.

(a) Show that $u$ is harmonic in $\mathbb{R}^2 \setminus \{(0,0)\}$.

(b) Show that there is no holomorphic function $f : \mathbb{C} \setminus \{0\} \rightarrow \mathbb{C}$ such that $u(x,y) = \text{Re} f(x + iy)$ for all $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$.

I.3 Consider the function $f : \mathbb{C} \rightarrow \mathbb{C}$ given by $f(z) := e^{z^2} - 4z^2$ for each $z \in \mathbb{C}$. Determine the number of zeroes (counted with multiplicities) of $f$ in the unit disk.

I.4 Let $f$ be an entire function such that $|f(z)| \geq |z|^k$ with $k \geq 1$ when $|z|$ is large enough. Show that $f$ is a polynomial of degree at least $k$.

Part II. Do two of these problems.

II.1 Let $D = \{z \in \mathbb{C} : |z| < 1\}$ and $K \subset D$ a compact subset. Suppose that $f : D \setminus K \rightarrow \mathbb{C}$ is holomorphic and that there is a sequence of polynomials $\{p_n\}_{n=1}^\infty$ such that $p_n \rightarrow f$ as $n \rightarrow \infty$ uniformly on compact subsets of $D \setminus K$. Show that there exists a holomorphic function $g : D \rightarrow \mathbb{C}$ such that $f = g$ on $D \setminus K$.

II.2 Let $G \subset \mathbb{C}$ be an open set, let $\{h_n\}_{n=1}^\infty$ be a uniformly bounded sequence of harmonic functions $h_n : G \rightarrow \mathbb{R}$, for each $n \in \mathbb{N}$, which is pointwise increasing.
a) Show that the pointwise defined limit function \( h : G \to \mathbb{R} \) given by 
\[
h(z) = \lim_{n \to \infty} h_n(z)
\]
for each \( z \in G \) is continuous.

b) Show that in fact \( h \) is harmonic in \( G \).

**Hint.** The mean value theorem for harmonic functions and Harnack's inequality (if \( u \geq 0 \) is harmonic in a neighborhood of the closure of the disc \( B(a, R) \), then 
\[
(R - r)u(a)/(R + r) \leq u(a + re^{i\theta}) \leq (R + r)u(a)/(R - r)
\]
for \( 0 \leq r < R \) and \( \theta \in \mathbb{R} \)) may be of use.

**II.3** Let \( D = \{ z \in \mathbb{C} : |z| < 1 \} \) and suppose that the function \( f \) is defined and continuous in 
\( A = D \cap \{ z : \text{Im}(z) \geq 0 \} \), holomorphic in 
\( D \cap \{ z : \text{Im}(z) > 0 \} \), real-valued on \( I = (-1, 1) \). Suppose there is a sequence \( \{x_n\}_{n=1}^\infty \subset I \) of distinct points tending to 0 with 
\[
f(x_n) - x_n^2 = 0.
\]
Show that \( f(z) = z^2 \) in \( A \).

__Note:__ Justify your answers thoroughly. For any theorem that you wish to cite, you should either give its name or a statement of the theorem. For each \( z_o \in \mathbb{C} \) and \( r > 0 \), the open ball in \( \mathbb{C} \) with center \( z_o \) and radius \( r \) is denoted by \( B(z_o, r) \).