Complex Analysis Ph.D. Qualifying Exam
Temple University
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• Justify your answers thoroughly.
• Notation: \( \mathbb{C} \) denotes the set of complex numbers. For each \( z_0 \in \mathbb{C} \) and \( r > 0 \) the open ball in \( \mathbb{C} \) with center \( z_0 \) and radius \( r \) is denoted by \( B(z_0, r) \).
• For any theorem that you wish to cite, you should either give its name or a statement of the theorem.

Part I. (Do 3 problems):

I.1. Let \( f : \mathbb{C} \rightarrow \mathbb{C} \) be an entire function such that \( \lim_{|z| \to \infty} \frac{f(z)}{z} = 0 \). Show that \( f \) is a constant.

I.2. Let \( u : \mathbb{R}^2 \rightarrow \mathbb{R} \) be a harmonic function. Assume that there exist sequences \( \{x_k\}_{k \in \mathbb{N}}, \{y_k\}_{k \in \mathbb{N}} \subseteq \mathbb{R} \) such that, if for each \( k \in \mathbb{N} \) we set \( z_k := x_k + iy_k \), then the sequence \( \{z_k\}_{k \in \mathbb{N}} \subseteq \mathbb{C} \) has pairwise distinct points, is convergent, and
\[
(\partial_1 u)(x_k, y_k) = (\partial_2 u)(x_k, y_k) = 0 \quad \text{for each } k \in \mathbb{N}.
\]
Show that the function \( u \) is constant.

I.3. Let \( z_0 \in \mathbb{C} \) and \( f : \mathbb{C} \setminus \{z_0\} \rightarrow \mathbb{C} \) be an analytic function. Show that if \( f \) has an essential singularity at \( z_0 \) then so does the function \( g : \mathbb{C} \setminus \{z_0\} \rightarrow \mathbb{C} \) given by \( g(z) := e^{f(z)} \) for each \( z \in \mathbb{C} \setminus \{z_0\} \).

I.4. Show that the equation \( z^5 + 15z + 1 = 0 \) has precisely four solutions in the annulus \( A := \{ z \in \mathbb{C} : \frac{3}{2} < |z| < 2 \} \).
Part II. (Do 2 problems):

II.1. Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be an entire one-to-one function. Show that there exist $a, b \in \mathbb{C}$ such that $a \neq 0$ and $f(z) = az + b$ for each $z \in \mathbb{C}$.

II.2. Let $a, b \in \mathbb{C}$ be such that $\text{Re } a > 0$, $\text{Re } b > 0$ and $a \neq b$. Evaluate

$$\int_{-\infty}^{\infty} \frac{x^3 e^{ix}}{(x^2 + a^2)(x^2 + b^2)} \, dx.$$ 

II.3. Let $r > 0$ and consider an analytic function $f : B(0, r) \rightarrow \mathbb{C}$ such that $f(0) = 0$ and there exists $M \in \mathbb{R}$, $M \geq 0$, with the property that $|f(z)| \leq M$ for each $z \in B(0, r)$. Show that

$$|f(z)| \leq \frac{M|z|}{r} \quad \forall \, z \in B(0, r).$$