Complex Analysis Ph.D. Qualifying Exam

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- Justify your answers thoroughly.
- Notation: $\mathbb{C}$ denotes the set of complex numbers. For each $z_o \in \mathbb{C}$ and $r > 0$ the open ball in $\mathbb{C}$ with center z_o and radius r is denoted by $B(z_o, r)$.
- For any theorem that you wish to cite, you should either give its name or a statement of the theorem.

Part I. (Do 3 problems):

I.1. Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be an entire function such that $\lim\limits_{|z|\to\infty} \dfrac{f(z)}{z} = 0$. Show that f is a constant.

I.2. Let $u : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a harmonic function. Assume that there exist sequences $\{x_k\}_{k\in\mathbb{N}}, \{y_k\}_{k\in\mathbb{N}} \subseteq \mathbb{R}$ such that, if for each $k \in \mathbb{N}$ we set $z_k := x_k + iy_k$, then the sequence $\{z_k\}_{k\in\mathbb{N}} \subseteq \mathbb{C}$ has pairwise distinct points, is convergent, and

$$(\partial_1 u)(x_k, y_k) = (\partial_2 u)(x_k, y_k) = 0 \quad \text{for each } k \in \mathbb{N}.$$

Show that the function u is constant.

I.3. Let $z_o \in \mathbb{C}$ and $f : \mathbb{C} \setminus \{z_o\} \longrightarrow \mathbb{C}$ be an analytic function. Show that if f has an essential singularity at z_o then so does the function $g : \mathbb{C} \setminus \{z_o\} \longrightarrow \mathbb{C}$ given by $g(z) := e^{f(z)}$ for each $z \in \mathbb{C} \setminus \{z_o\}$.

I.4. Show that the equation $z^5 + 15z + 1 = 0$ has precisely four solutions in the annulus $A := \left\{z \in \mathbb{C} : \frac{3}{2} < |z| < 2\right\}$.

Part II. (Do 2 problems):

II.1. Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be an entire one-to-one function. Show that there exist $a, b \in \mathbb{C}$ such that $a \neq 0$ and $f(z) = az + b$ for each $z \in \mathbb{C}$.

II.2. Let $a, b \in \mathbb{C}$ be such that $\operatorname{Re} a > 0$, $\operatorname{Re} b > 0$ and $a \neq b$. Evaluate

$$\int_{-\infty}^{\infty} \frac{x^3 e^{ix}}{(x^2 + a^2)(x^2 + b^2)}\, dx.$$

II.3. Let $r > 0$ and consider an analytic function $f : B(0, r) \longrightarrow \mathbb{C}$ such that $f(0) = 0$ and there exists $M \in \mathbb{R}$, $M \geq 0$, with the property that $|f(z)| \leq M$ for each $z \in B(0, r)$. Show that

$$|f(z)| \leq \frac{M|z|}{r} \qquad \forall\, z \in B(0, r).$$