Part I: Do three of the following problems

1. Let $u(x, y)$ be a polynomial of degree $n$ that is harmonic in $\mathbb{C}$. Show that $u(x, y) = \Re(f(z))$ where $f(z)$ is a polynomial of degree $n$.

2. Let $f(z)$ be an analytic function from the open unit disc $D$ onto $D$. Let $M(r) = \max\{|f(z)| : |z| = r\}$.
   (i) Show that $M(r)$ is a strictly increasing function of $r$.
   (ii) Show that $\lim_{r \to 1^-} M(r) = 1$.

3. Use the calculus of residues to find the principal value of $\int_{-\infty}^{\infty} \frac{dx}{x(2x^2 - 2x + 1)}$. Here the principal value of $\int_{-\infty}^{\infty} f(x)dx$ means $\lim_{\epsilon \to 0^+} \left(\int_{-\infty}^{-\epsilon} f(x)dx + \int_{\epsilon}^{\infty} f(x)dx\right)$.

4. (i) Find a bijective conformal mapping from $\mathbb{C} - [1, \infty)$ to the open unit disc $D$.
   (ii) Find a conformal mapping from $\mathbb{C} - [0, 1]$ onto the open unit disc $D$. Can this map be bijective? Why or why not?
Part II: Do two of the following problems

1. Let \( G \subset \mathbb{C} \) be a region in \( \mathbb{C} \) and let \( I = [a, b] \) be a line segment, \( I \subset G \). Let \( f(z) \) be continuous in \( G \) and analytic in \( G - I \). Show that \( f(z) \) is analytic in \( G \).

2. Let \( f(z) \) be analytic in \( \{z : 0 < |z| < 1\} \) except for a sequence of isolated non removable singularities \( \{z_n\} \) with \( \lim_{n \to \infty} z_n = 0 \). Show that any \( w \in \mathbb{C} \) and any \( \epsilon, \delta > 0 \), there exists a \( z \neq z_n \) with \( 0 < |z| < \delta \) such that \( |f(z) - w| < \epsilon \).

3. Let \( G \) be a region in \( \mathbb{C} \) that contains the closed unit disc \( \bar{D} \) and let \( f(z, t) \) be a continuous function on \( G \times [0, 1] \) that is analytic in \( z \).

   (i) Show that \( f'(z, t) \) is continuous on \( G \times [0, 1] \), where \( f'(z, t) \) denotes \( \frac{\partial}{\partial z} f(z, t) \).

   (ii) Suppose \( f(z, t) \neq 0 \) for \( z \) with \( |z| = 1 \) and any \( t \in [0, 1] \). Show that \( f(z, 1) \) has the same number of zeroes in \( D \), counting multiplicities, as \( f(z, 0) \).