Part I: Do three of the following problems

1. Let $f(z)$ be an entire function. Suppose there exists $z_0 \in \mathbb{C}$ and $R > 0$ such that $B(z_0; R) \cap f(\mathbb{C}) = \emptyset$, where $B(z_0; R) = \{z \in \mathbb{C} : |z - z_0| < R\}$. Show that $f(z)$ is a constant function.

2. Let $U$ be the open unit disc and let $f(z)$ be a nonconstant function analytic on some open set containing $\bar{U}$. Suppose $f(\partial U) \subset \partial U$. Prove that
   (a) $f(U) \subset U$;
   (b) $f(z)$ has a zero in $U$.

3. Use the residue theorem to evaluate $\int_0^\infty \frac{\cos x - 1}{x^2(x^2 + 1)} \, dx$.

4. Suppose $f(z)$ has a pole at $z = a$.
   (a) Prove that for any $\delta > 0$ there exists an $R > 0$ such that
       $$\{z \in \mathbb{C} : |z| > R\} \subset f(\text{ann}(a; 0, \delta)),$$
   where $\text{ann}(a; 0, \delta)) = \{z \in \mathbb{C} : 0 < |z - a| < \delta\}$.
   (b) Prove that $e^{f(z)}$ has an essential singularity at $z = a$. 


Part II: Do two of the following problems

1. Let $G$ be a simply connected domain and let $f(z)$ be analytic in $G$.
   (a) Prove that there exists a function $F(z)$ analytic in $G$ such that $F'(z) = f(z)$.
   (b) Suppose further that $f(z) \neq 0$ for any $z \in G$. Prove that there exists a function $g(z)$ analytic in $G$ such that $f(z) = e^{g(z)}$ and a function $h(z)$ analytic in $G$ such that $f(z) = (h(z))^3$.

2. (a) Prove that $\cos \pi z = \prod_{n=0}^{\infty} \left(1 - \frac{4z^2}{(2n+1)^2}\right)$.
   (b) Prove that $\pi \tan \pi z = -\sum_{n=0}^{\infty} \frac{8z}{4z^2 - (2n+1)^2}$ for every $z \neq n + \frac{1}{2}$, $n \in \mathbb{Z}$

3. Let $\{f_n(z)\}$ be a sequence of analytic functions in $G$. Suppose $\{f_n(z)\}$ converges to a function $f(z)$ uniformly on every compact subset of $G$ and that $f(z)$ has no zeros in $G$.
   (a) Let $B(a; R)$ be an open disc such that $\bar{B}(a; R) \subset G$. Prove that there exists an $N > 0$ such for any $n > N$ $f_n(z)$ has no zeros in $B(a; R)$.
   (b) Let $K$ be a compact subset of $G$. Prove that there exists an $N > 0$ such for any $n > N$ $f_n(z)$ has no zeros in $K$.
   (c) Let $S = \{z \in G : f_n(z) = 0 \text{ for some } n \in \mathbb{N}\}$. Prove that $S$ has no accumulation points in $G$.  