Ph.D. Comprehensive Examination
Complex Analysis Section
August 2009

Part I. Do three of these problems.

I.1. Compute
$$\int_0^\infty \frac{x^2}{1+x^2+x^4}\,dx$$
using residues.

I.2. Suppose that D is a region and that $f : D \to \mathbb{C}$ is a one-to-one analytic function. Show that $f'(z) \neq 0$ for every $z \in D$.

I.3. Suppose $f(z)$ is meromorphic in all of $\mathbb{C}$ and such that
$$|f(z)| \leq C|z|^m \quad \text{on } \{z : |z| > R\}$$
for some C and $R > 0$. Prove that $f(z)$ is rational.

I.4. Let $G \subset \mathbb{C}$ be open and $\mathcal{H}(G)$ the set of holomorphic functions on G. Show that $\mathcal{H}(G)$ has the property
$$f, g \in \mathcal{H}(G) \text{ and } fg \equiv 0 \implies f \equiv 0 \text{ or } g \equiv 0$$
if and only if G is connected.

Part II. Do two of these problems.

II.1. Let $G \subset \mathbb{C}$ be open, $a \in G$, and
$$\mathcal{F} = \{f : G \to \mathbb{C} : f \text{ is holomorphic and } |f(z)| \leq 1 \ \forall z \in G\}.$$
Let $\phi : \mathcal{F} \to \mathbb{R}$ be defined by $\phi(f) = |f''(a)|$. Show that ϕ has a maximum: there is $f_0 \in \mathcal{F}$ such that $\phi(f_0) = \sup\{\phi(f) : f \in \mathcal{F}\}$.

II.2. Let $p(z)$ and $q(z)$ be polynomials of the same degree $n > 0$, let $f(z)$ be an entire function. Suppose $q(z) = p(f(z))$ for all z. Show that $f(z) = az + b$ for some a and b. Hint: Show first that f must be a polynomial.

II.3. Let $p(z)$ be a polynomial of degree n. Suppose that $G = \{z : |p(z)| < 1\}$ is simply connected with, say, C^1 boundary. Let $D = \{z : |z| < 1\}$ and let $f : D \to G$ be biholomorphic such that it and its inverse are continuous up to the boundary. Let $\alpha_1, \ldots, \alpha_n$ be the zeros of $p(z)$ counting multiplicity. Let $a_j = f^{-1}(\alpha_j)$ and define
$$h(z) = \frac{a_1 - z}{1 - \overline{a}_1 z} \cdots \frac{a_n - z}{1 - \overline{a}_n z}$$
Show that there is θ such that
$$h(z) = e^{i\theta} p(f(z)) \quad \text{for } z \in D.$$
Hint: Consider the zeros of h and $p \circ f$. What is $|h(z)|$ when $z \in \partial D$?