Ph.D. Comprehensive Examination in Complex Analysis
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Part I: Do three of the following problems

1. Let $U(x, y)$ be a harmonic function on $\mathbb{C}$. Suppose $u(x, y)$ is harmonic on $\mathbb{C}$, and let $v(x, y)$ be its harmonic conjugate. Show that $U(u(x, y), v(x, y))$ is harmonic.

2. Find a bijective holomorphic mapping that maps the set $\{z \in \mathbb{C} : |z| < 1, Im(z) > 0\}$ onto the open unit disk.

3. Let C be a simple closed contour (C has counterclockwise orientation) enclosing the points $0, 1, ..., k$ in the complex plane, where k is a positive integer. Let

$$I_k = \int_C \frac{1}{z(z-1)...(z-k)}\,dz \quad \text{and} \quad J_k = \int_C \frac{(z-1)...(z-k)}{z}\,dz$$

Show that $J_k = 2\pi i(-1)^k k!$ and that $I_k = 0$. Hint: to show that $I_k = 0$ replace C by a circle $|z| = R$ for R sufficiently large (be sure to explain why this does not affect the value of the integral) and let R go to ∞.

4. Let $f(z)$ be an analytic function from the open unit disk to itself. Suppose that $f(1/2) = 0$. Use Schwarz's lemma to prove that $|f(0)| \leq 1/2$ and $|f'(1/2)| \leq 4/3$.

Part II: Do two of the following problems

1. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. Suppose f maps every unbounded sequence to an unbounded sequence. Prove that f is a polynomial.

2. (a) Let $\{f_n(z)\}$, $f_n(z) = \sum_{k=0}^{\infty} a_{kn} z^k$, be a sequence of analytic functions on the open unit disk D. Suppose $\{f_n(z)\}$ converges to a function $f(z)$ in D; moreover, suppose that the convergence is uniform on every set $\{z \in D : \ |z| \leq r\}$, where $r < 1$. Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$. Show that for every $k \geq 0$, $\lim_{n\to\infty} a_{kn} = a_k$.

(b) Conversely, suppose $\{f_n(z)\}$, $f_n(z) = \sum_{k=0}^{\infty} a_{kn} z^k$ is a sequence of analytic functions on the open unit disk D. Suppose that for every $k \geq 0$ $\lim_{n\to\infty} a_{kn}$ exists and equals a_k . Suppose further that there exists $M > 0$ such that $|f_n(z)| \leq M$ for every $n \geq 1$ and every $z \in D$. Show that $\{f_n(z)\}$ converges to $f(z) = \sum_{k=0}^{\infty} a_k z^k$ in D and that the convergence is uniform on every set $\{z \in D : \ |z| \leq r\}$, where $r < 1$.

3. Compute

$$\mathcal{S}(L)(\omega, \xi, h) = \int_0^{\infty} L(\omega, u + ih) \sin(u\xi) du,$$

where $h > 0$, $\omega > 0$, $\xi > 0$ and

$$L(\omega, q) = \frac{1}{\omega + q} + \frac{1}{\bar{q} - \omega}$$