# Ph.D. Comprehensive Examination in Complex Analysis
# Department of Mathematics, Temple University

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**Part I: Do three of the following problems**

1. Show that the function $u(x,y) = \log(x^2 + y^2)$ is harmonic in $\mathbb{C} \setminus \{0\}$ but that it has no harmonic conjugate in $\mathbb{C} \setminus \{0\}$.

2. Let $f(z)$ be an entire function.

(a) Suppose $\int_0^{2\pi} |f(re^{i\theta})| \, d\theta \leq r^{17/3}$ for all $r$ sufficiently large. Prove that $f(z)$ is a polynomial of degree at most 5.

(b) Suppose that $\int_0^{2\pi} |f(re^{i\theta})| \, d\theta \leq r^{17/3}$ for all $r \geq 0$. Prove that $f(z) \equiv 0$.

3. Let $f(t)$ be a continuous real-valued function on the interval $[0,1]$. Set $h(z) = \int_0^1 f(t)e^{zt} \, dt$.

(a) Prove that $h(z)$ is an entire function.

(b) Let $h(z) = \sum_{n=0}^{\infty} a_n z^n$. Show that $a_n = \dfrac{1}{n!} \int_0^1 f(t)t^n \, dt$.

(c) Prove that if $h(z) \equiv 0$, then $f(t) \equiv 0$.

4. Evaluate $\displaystyle\int_0^{\infty} \frac{x - \sin x}{x^3} \, dx$.

**Part II: Do two of the following problems**

1. Let $U$ and $V$ be two open connected subsets of $\mathbb{C}$ and let $f : U \to V$ be an analytic nonconstant function on $U$. Suppose that for any $K \subset V$ compact $f^{-1}(K)$ is compact. Show that $f(U) = V$.

2. Suppose $f(z)$ is analytic on the right half-plane $H = \{z : \operatorname{Re}(z) > 0\}$ and satisfies $|f(z)| \leq 1$ for all $z \in H$, $f(1) = 0$. Find the largest possible value of $|f'(1)|$ and determine all functions $f(z)$ for which $|f'(1)|$ is the largest possible.

3. (a) Let $f(z)$ be analytic in a region $G$ that contains the closed ball $\bar{B}(0, R)$. Let $f_n(z)$ be a sequence of analytic functions on $G$ that converges uniformly to $f(z)$. Suppose further that $f(z) \neq 0$ for any $z$ with $|z| = R$. Show that for all $n$ sufficiently large $f_n(z)$ has the same number of zeros in the open ball $B(0, R)$ as $f(z)$.

(b) Determine with proof the number of zeros of $\displaystyle\sum_{n=0}^{N}(-1)^n \frac{z^{2n+1}}{(2n+1)!}$ in the open ball $B(0, 5\pi/2)$ for all $N$ sufficiently large.