PH.D. COMPREHENSIVE EXAMINATION
COMPLEX ANALYSIS SECTION

August, 2006

Part I. Do three (3) of these problems.

I.1. (a) Let \( f(z), g(z) \) be entire functions and assume that \( f(z) = g(z) \) for every \( z \in S \) where \( S \) is an uncountable subset of \( \mathbb{C} \). What can be said about \( f \) and \( g \)?

(b) Let \( \Omega \) be a region and \( f, g \) holomorphic in \( \Omega \) such that \( f(z)g(z) = 0 \) for all \( z \in \Omega \). Show that either \( f \equiv 0 \) or \( g \equiv 0 \).

I.2. Let \( n \) be a positive integer. Prove that the polynomial

\[
p(z) = \sum_{k=0}^{n} \frac{z^k}{k!} = 1 + z + \frac{z^2}{2} + \cdots + \frac{z^n}{n!}
\]

has \( n \) distinct zeros.

I.3. Assume \( f \) is entire and \( |f(z)| \to +\infty \) as \( |z| \to +\infty \). Prove that \( f \) is a polynomial.

I.4. Determine the group of all one-to-one holomorphic maps of \( \mathbb{C} \) onto \( \mathbb{C} \).
Part II. Do two (2) of these problems.

II.1. Suppose \( f : \mathbb{U} \longrightarrow \mathbb{U} \) is nonconstant, holomorphic on \( U \) and continuous on \( \mathbb{U} \) (\( U \) denotes the unit disc). Assume \( f(\partial U) \subseteq \partial U \). Prove that
(a) \( f \) has a zero in \( U \).
(b) \( f \) maps the disc \( U \) onto itself.

II.2. Give an example of a function which is holomorphic at every point of the complex plane except for a single point on the unit circle \( |z| = 1 \), and which is real-valued at every other point of the unit circle.

II.3. Suppose \( f \) is holomorphic in the half plane \( \text{Im} \ z > 1 \) and \( f(z + 1) = f(z) \).
(a) Use a Laurent expansion to prove that \( f(z) \) has an exponential representation valid for \( \text{Im} \ z > 1 \):

\[
(A) \quad f(z) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi inz}.
\]

(b) Derive a special form of (A) that holds if \( |f(z)| \) remains bounded as \( \text{Im} \ z \to \infty \).