PH.D. COMPREHENSIVE EXAMINATION
COMPLEX ANALYSIS SECTION

August, 2006

Part I. Do three (3) of these problems.

I.1. (a) Let $f(z)$, $g(z)$ be entire functions and assume that $f(z) = g(z)$ for every $z \in S$ where S is an uncountable subset of $\mathbb{C}$. What can be said about f and g?
(b) Let Ω be a region and f, g holomorphic in Ω such that $f(z)g(z) = 0$ for all $z \in \Omega$. Show that either $f \equiv 0$ or $g \equiv 0$.

I.2. Let n be a positive integer. Prove that the polynomial

$$p(z) = \sum_{k=0}^{n} \frac{z^k}{k!} = 1 + z + \frac{z^2}{2} + \cdots + \frac{z^n}{n!}$$

has n distinct zeros.

I.3. Assume f is entire and $|f(z)| \to +\infty$ as $|z| \to +\infty$. Prove that f is a polynomial.

I.4. Determine the group of all one-to-one holomorphic maps of $\mathbb{C}$ onto $\mathbb{C}$.

Part II. Do two (2) of these problems.

II.1. Suppose $f : \overline{U} \longrightarrow \overline{U}$ is nonconstant, holomorphic on U and continuous on $\overline{U}$ (U denotes the unit disc). Assume $f(\partial U) \subseteq \partial U$. Prove that

(a) f has a zero in U.
(b) f maps the disc U onto itself.

II.2. Give an example of a function which is holomorphic at every point of the complex plane except for a single point on the unit circle $|z| = 1$, and which is real-valued at every other point of the unit circle.

II.3. Suppose f is holomorphic in the half plane $\operatorname{Im} z > 1$ and $f(z+1) = f(z)$.
(a) Use a Laurent expansion to prove that $f(z)$ has an exponential representation valid for $\operatorname{Im} z > 1$:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n z} \tag{A}$$

(b) Derive a special form of (A) that holds if $|f(z)|$ remains bounded as $\operatorname{Im} z \to \infty$.