August 25, 2004

Comprehensive Examination Department of Mathematics

Complex Analysis

Part I: Do three of the following problems

1. Suppose $f(z)$ and $\overline{f(z)}$ are both analytic in an open connected subset S of $\mathbf{C}$. Show that $f(z)$ is constant in S.

2. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function. Suppose $f(z)$ maps the real axis to the real axis and the imaginary axis to the imaginary axis. Prove that $a_n = 0$ for all n even.

3. Let $f(z)$ be analytic in $\mathbf{C} \setminus \{\pm 1\}$. Suppose there exist $A, B > 0$ such that for any $z \in \mathbf{C} \setminus \{\pm 1\}$,

$$|f(z)| \leq \frac{A}{|z-1|} + \frac{B}{|z+1|}.$$

Show that $f(z) = \frac{p(z)}{z^2-1}$ where $p(z)$ is a polynomial of degree less or equal to 1.

4. Use the calculus of residues to find

$$\int_{-\infty}^{\infty} \frac{1}{x^2+2x+2}\,dx \text{ and } \int_{-\infty}^{\infty} \frac{1}{(x^2+2x+2)^4}\,dx.$$

Part II: Do two of the following problems

1. Let $f(z)$ be analytic on an open set G that contains the closed unit disc $\bar{D} = \{z \in \mathbf{C} : |z| \leq 1\}$, and let $\{f_n(z)\}$ be a sequence of analytic functions that converges to $f(z)$ uniformly in G. Suppose that for any z with $|z| = 1$, $f(z) \neq 0$. Show that for all n sufficiently large $f_n(z)$ has the same number of zeros in $\bar{D}$ as $f(z)$.

2. Let $H = \{z \in \mathbf{C} : Im(z) > 0\}$ be the upper half-plane, and let $f : H \to H$ be an analytic function such that $f(i) = i$.

(a) Show that $|f'(i)| \leq 1$.

(b) Show that if $|f'(i)| = 1$, then $f(z)$ is a Möbius transformation.

3. Show that any function $f(z)$ meromorphic in $\mathbf{C}$ can be written as $f(z) = \frac{h(z)}{g(z)}$ where $h(z)$ and $g(z)$ are entire functions with $|h(z)| + |g(z)| > 0$ for any $z \in \mathbf{C}$.