August 25, 2004

Comprehensive Examination Department of Mathematics

Complex Analysis

Part I: Do three of the following problems

1. Suppose \( f(z) \) and \( \overline{f(z)} \) are both analytic in an open connected subset \( S \) of \( \mathbb{C} \). Show that \( f(z) \) is constant in \( S \).

2. Let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) be an entire function. Suppose \( f(z) \) maps the real axis to the real axis and the imaginary axis to the imaginary axis. Prove that \( a_n = 0 \) for all \( n \) even.

3. Let \( f(z) \) be analytic in \( \mathbb{C} \setminus \{ \pm 1 \} \). Suppose there exist \( A, B > 0 \) such that for any \( z \in \mathbb{C} \setminus \{ \pm 1 \} \),

\[
|f(z)| \leq \frac{A}{|z - 1|} + \frac{B}{|z + 1|}.
\]

Show that \( f(z) = \frac{p(z)}{z^2 - 1} \) where \( p(z) \) is a polynomial of degree less or equal to 1.

4. Use the calculus of residues to find

\[
\int_{-\infty}^{\infty} \frac{1}{x^2 + 2x + 2} \, dx \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{1}{(x^2 + 2x + 2)^4} \, dx.
\]
Part II: Do two of the following problems

1. Let $f(z)$ be analytic on an open set $G$ that contains the closed unit disc $D = \{ z \in \mathbb{C} : |z| \leq 1 \}$, and let $\{f_n(z)\}$ be a sequence of analytic functions that converges to $f(z)$ uniformly in $G$. Suppose that for any $z$ with $|z| = 1$, $f(z) \neq 0$. Show that for all $n$ sufficiently large $f_n(z)$ has the same number of zeros in $\bar{D}$ as $f(z)$.

2. Let $H = \{ z \in \mathbb{C} : \text{Im}(z) > 0 \}$ be the upper half-plane, and let $f : H \to H$ be an analytic function such that $f(i) = i$.
   (a) Show that $|f'(i)| \leq 1$.
   (b) Show that if $|f'(i)| = 1$, then $f(z)$ is a Möbius transformation.

3. Show that any function $f(z)$ meromorphic in $\mathbb{C}$ can be written as $f(z) = \frac{h(z)}{g(z)}$ where $h(z)$ and $g(z)$ are entire functions with $|h(z)| + |g(z)| > 0$ for any $z \in \mathbb{C}$. 