Ph.D. Comprehensive Examination
Complex Analysis
Fall 2000

Part I. Do three of these problems.

I.1. Show that if \( u(x, y) \) is a harmonic function in a simply connected region, \( D \) then, then \( u \) is the real part of a function that is analytic in \( D \).

I.2. Let \( f(z) \) be analytic on \( \{ z : 0 < |z| < 2 \} \) and suppose that for \( n = 0, 1, 2, \ldots \)

\[
\int_{|z|=1} z^n f(z) \, dz = 0.
\]

Show that \( f \) has a removable singularity at \( z = 0 \).

I.3. Suppose that \( f(z) \) is an entire function satisfying \( |f(z)| > 1 \) when \( |z| > 1 \). Prove that \( f(z) \) is a polynomial.

I.4. Suppose that \( f : \mathbb{C} \to \mathbb{C} \) is entire and that \( f \) has exactly \( k \) zeros in the open disc \( \{ z : |z| < 1 \} \) but none on the circle \( \{ z : |z| = 1 \} \). Show that there exists \( \varepsilon > 0 \) such that any entire function \( g \) that satisfies \( |f(z) - g(z)| < \varepsilon \) on the circle \( |z| = 1 \) must also have exactly \( k \) zeros in the open disc \( \{ z : |z| < 1 \} \).
Part II. Do two of these problems.

II.1. Let $\alpha > 0$. Prove that

1. if $f \in L^1(0,1)$ then
   \[ f_\alpha(x) = \int_0^x (x-t)^{\alpha-1} f(t) \, dt \]
   exists a.e. and is integrable on $(0,1)$;

2. if $f \in L^p(0,1)$ then $f_\alpha$ is continuous in $(0,1)$ for $\alpha > 1/p$.

II.2. Let $1 \leq p, q \leq \infty$, $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$. Prove that $fg \in L^r(\mathbb{R}^n)$ with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$.

II.3. Prove that

1. \[ \log \frac{1}{1-x} = \sum_{n=1}^{\infty} \frac{x^n}{n}. \]

2. \[ \int_0^1 \log \frac{1}{1-x} \, dx = 1. \]