August, 1998

Comprehensive Examination **Department of Mathematics**

COMPLEX ANALYSIS

PART I: Do three of the following problems.

1. (a) Find

$$Res[\frac{e^{iz}}{(z^2+1)^5}; i].$$

   (b) Evaluate

$$\int_0^\infty \frac{cos x}{(x^2+1)^5} dx.$$

2. Let $u(x, y)$ be an everywhere positive harmonic function on $\mathbf{C}$. Prove that $u(x, y)$ is constant.

3. Show that if $f(z)$ is analytic at α and

$$g(z) = \frac{f(z) + \alpha f'(\alpha) - z f'(\alpha) - f(\alpha)}{(z-\alpha)^2},$$

   then $g(z)$ has a removable singularity at $z = \alpha$.

4. Find an entire function having a zero of order n at $z = n$, $n = 1, 2, 3, ...$, and no other zeros.

PART II: Do two of the following problems.

1. Suppose $f(z)$ is an entire function with the property that for every $w \in \mathbf{C}$ the equation $f(z) = w$ has precisely k solutions. Show that $f(z)$ is a polynomial of degree k.

2. Suppose $\{f_n\}$ is a sequence of analytic functions on a region D such that there exists a positive constant M with the property that

$$\int\int_D |f_n(z)|^2 dxdy \leq M \text{ for all n.}$$

Show that $\{f_n\}$ has subsequence that converges uniformly on compact subsets of D.

Hint: If f is analytic in a neighborhood of a closed ball $\overline{B(a;R)}$, show that

$$|f(a)|^2 \leq \frac{1}{\pi R^2} \int_0^{2\pi} \int_0^R |f(a + re^{i\theta})|^2 r dr d\theta.$$

3. Suppose $f(z)$ is analytic on $|z| < 1$ and continuous on $|z| \leq 1$. Assume $f(z) = 0$ on an arc of the circle $|z| = 1$. Prove that $f(z) \equiv 0$.