PH.D. COMPREHENSIVE EXAMINATION
COMPLEX ANALYSIS SECTION

Fall 1995

Part I. Do three (3) of these problems.

I.1. Show that if \( u(x, y) + iv(x, y) \) is an analytic function with non-vanishing derivative in a region \( R \), then, for any constants \( c_1 \) and \( c_2 \), the curves \( u(x, y) = c_1 \) and \( v(x, y) = c_2 \) are orthogonal in \( R \) (at the points of their intersection).

I.2. If \( -1 < a < 1 \), compute

\[
\int_{0}^{\infty} \frac{x^a}{1 + x^2} dx
\]

using residues.

I.3. Give a conformal (i.e., biholomorphic) map of \( \mathbb{C} \setminus [1, \infty) \) onto the open unit disc.

I.4. Suppose \( f(z) \) is holomorphic in \( \mathbb{C} \setminus \{0\} \) and satisfies

\[
|f(z)| \leq |z|^2 + \frac{1}{|z|^2} \quad \text{for} \quad z \neq 0.
\]

If \( f(z) \) is an odd function, what form must it have?

Part II. Do two (2) of these problems.

II.1. Suppose \( f(z) \) is meromorphic in all of \( \mathbb{C} \) and bounded on \( \{z : |z| > R\} \) for some \( R > 0 \). Prove that \( f(z) \) is rational.

II.2. Suppose \( f \) is analytic on a neighborhood of the closed unit disc \( \overline{D} \) and one-to-one on the unit circle \( \partial D \). Show that \( f \) is one-to-one on \( \overline{D} \).

II.3. Show that there is no one-to-one analytic function which maps \( A = \{z : 0 < |z| < 1\} \) onto \( B = \{z : 1 < |z| < 2\} \).