Applied Math Exam

PART I. Do three of the following four problems.

1. Approximate all roots of the equation

$$\sin(x^2 + \epsilon \sin x) = 0$$

to $O(\epsilon^2)$ error.

2. Find the extremal curve for the functional

$$J[y] = \int_0^1 (1 + y_{xx}^2) dx$$

subject to the boundary conditions $y(0) = 0$, $y'(0) = 1$, and $y(1) = 1$, $y'(1) = 1$, and

3. Consider the equation

$$\frac{d^2}{dx^2} y(x) + k y(x) = f(x), \;\; 0 < x < \pi, \;\; y(0) = y(\pi) = 0$$

with

$$f(x) = \begin{cases} 1 & 0 < x \leq \pi/2 \\ -1 & \pi/2 < x < \pi \end{cases} .$$

   (a) Show that there is no solution if $k = 4$.

   (b) Find the solution if $k = 1$.

4. The 3D wave equation

$$u_{tt} = c^2 \nabla^2 u$$

under the assumption of spherical symmetry, i.e., $u(\mathbf{x}, t) = u(|\mathbf{x}|, t)$. reduces to

$$\frac{1}{c^2} u_{tt} = u_{rr} + \frac{2}{r} u_r. \tag{1}$$

   (a) Find the general solution of (1), including incoming and outgoing waves.

   (b) Keeping only the outgoing waves and assuming the flux condition

$$0 < \lim_{r \to 0} r^2 u_r < \infty$$

   for all t, how does the amplitude of the wave front at $r = ct$ decay in time?

PART II. Do two of the following three problems.

1. Consider the system

$$\begin{aligned} \dot{x} &= -y + \mu x - x^2 - 4y^4, \\ \dot{y} &= x + \mu y - 4x^4 - 2y^2, \end{aligned}$$

with parameter μ. Find all bifurcations and sketch the bifurcation diagram (in μ).

2. Consider the equation

$$u_t + uu_x = \alpha u_{xx} + \beta u_{xxx}$$

with $\alpha > 0$ and $\beta > 0$.

   (a) Find an equation for traveling wave solutions of the form $u(x,t) = f(x - ct)$ with $c > 0$ and $\lim_{z\to\infty} f(z) = 0$. Find and classify all stationary points.

   (b) Find all possible values of $\lim_{z\to\infty} f(z)$

3. Consider the equation

$$\ddot{\theta} + \sin\theta = 0, \quad \theta(0) = \epsilon, \quad \dot{\theta}(0) = 0.$$

   (a) Show that the solution $\theta(t)$ is periodic in time.

   (b) Find the asymptotically correct period to $O(\epsilon^2)$ error.