January 2022 Applied Mathematics Qualifying Written Exam

PART I. Do three of the following four problems.

1. Compute a two term asymptotic approximation in $\epsilon$ for the solution of the equation
   \[
   \frac{d^2 h}{dt^2} = -\frac{1}{(1 + \epsilon h)^2}, \quad h(0) = 0, \quad \frac{dh}{dt}(0) = 1.
   \]

2. Given the system
   \[
   \frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}
   \]
   (a) Compute the eigenvalues and eigenvectors of the system.
   (b) Sketch the phase portrait.

3. Nondimensionalize and solve the problem
   \[
   k \nabla^2 u = -1
   \]
   on the domain $\{x^2 + y^2 < 1\}$ with boundary conditions $u = 0$ on $x^2 + y^2 = 1$.

4. Let
   \[X = \{ y \in C^2([0, 1]) : y(0) = y(1) = 0 \},\]
   and define a functional $J : X \to \mathbb{R}$ by
   \[
   J[y] = \int_0^1 xy \, dx.
   \]
   Find the minimizer of $J$ on $X$ subject to the constraint
   \[
   \int_0^1 (y')^2 \, dx = 1
   \]
PART II. Do two of the following three problems.

1. Determine the leading order approximation and first correction for all roots of the equation
   \[ \epsilon x^5 - x^3 - 1 = 0. \]

2. Find the first variation of the functional
   \[ J(y) = (y(2))^2 + \int_1^2 (xy + (y')^2) \, dx \]
   with admissible set \( \{ y \in C^2[1,2], y(1) = 3 \} \). What is the boundary value problem for its extrema? (Do not solve.)

3. (a) Find the adjoint operator for the operator
   \[ Lu = u_{xx} + 4\pi^2 u, \quad u(0) = u(1) = 0. \]

   (b) Find conditions on the parameters \( \alpha \) and \( \beta \) for which a solution to the problem
   \[ u_{xx} + 4\pi^2 u = f(x), \quad u(0) = u(1) = 0 \]
   exists, where \( f(x) = \alpha \sin(2\pi x) - \beta \sin^3(2\pi x) \).