1 Part I: do 3 of 4

1. Find all scaling transformations that leave the KdV equation

\[ u_t + 6uu_x + u_{xxx} = 0 \]

invariant.

2. What is the appropriate inner product for the Sturm-Liouville eigenvalue problem for Chebyshev’s polynomials?

\[(1 - x^2)y'' - xy' = \lambda y, \quad x \in (-1, 1)\]

3. Write the the first and second variations of the functional

\[ F[u] = \int_0^1 (x^4 - u(x)^6)^2 u'(x)^4 dx \]

4. At what boundary point does the solution of the boundary value problem

\[
\begin{cases}
\epsilon u'' + 2u' - 3u = 0, \\
u(0) = 1, \\
u(1) = 2;
\end{cases}
\]

develop a boundary layer as \( \epsilon \to 0^+ \)? Explain your answer. (You don’t have to find the actual solution.)


2 Part II: do 2 of 3

1. Consider the equation of motion of a uniform linearly elastic string under tension $T_0$, occupying at rest the interval $[0, L]$ along the $x$-axis in the $(x, y)$-plane

$$\rho \ddot{r} = \frac{\partial}{\partial x} ((T_0 + E\varepsilon) \tau), \quad \varepsilon = \left| \frac{\partial \mathbf{r}}{\partial x} \right| - 1, \quad \tau = \frac{\partial \mathbf{r}}{\partial x} / \left| \frac{\partial \mathbf{r}}{\partial x} \right|$$

where $\mathbf{r}(x, t)$ is the position vector of the material point $x$ at time $t$, $\rho$ (linear density) and $E$ (the Young’s modulus) are constant. Write the linearized equations for $u(x, t) = (u_1(x, t), u_2(x, t))$, assuming that $\mathbf{r}(x, t) = (x, 0) + \epsilon \mathbf{u}$ and $|\epsilon|$ is small. Show all work.

2. Consider the problem of minimizing the perimeter

$$\int_0^1 \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2} dt$$

of a simple closed curve with parametric equations

$$\begin{cases} x = x(t), \\
y = y(t) \\
t \in [0, 1] \end{cases}$$

subject to the area constraint

$$2A = \int_0^1 (x(t)\dot{y}(t) - y(t)\dot{x}(t)) dt.$$ 

Observe that the rotations $R_\theta$ through any angle $\theta$ leave the Lagrangian

$$L(x, y, \dot{x}, \dot{y}) = \sqrt{\dot{x}^2 + \dot{y}^2} - \lambda (x\dot{y} - y\dot{x})$$

invariant. Use Noether’s theorem to find the conserved quantity corresponding to this symmetry. Show all work.

3. Consider the initial value problem

$$\begin{cases} \ddot{x} - \epsilon(1 - x^2)\dot{x} + x = 0, \\
x(0) = 0, \\
\dot{x}(0) = 1. \end{cases}$$

when $|\epsilon|$ is small. Apply the two-scale expansion method to obtain uniform in time $O(\epsilon)$ approximation to the solution of the initial value problem above.