Applied Mathematics Qualifying Written Exam (January 2017)

1 Part I: do 3 of 4

1. Find all scaling transformations that leave the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0$$

invariant.

2. What is the appropriate inner product for the Sturm-Liouville eigenvalue problem for Chebyshev's polynomials?

$$(1-x^2)y'' - xy' = \lambda y, \quad x \in (-1,1)$$

3. Write the the first and second variations of the functional

$$F[u] = \int_0^1 (x^4 - u(x)^6)^2 u'(x)^4 dx$$

4. At what boundary point does the solution of the boundary value problem

$$\begin{cases} \epsilon u'' + 2u' - 3u = 0, \\ u(0) = 1, \\ u(1) = 2; \end{cases}$$

develop a boundary layer as $\epsilon \to 0^+$? Explain your answer. (You don't have to find the actual solution.)

2 Part II: do 2 of 3

1. Consider the equation of motion of a uniform linearly elastic string under tension T_0, occupying at rest the interval $[0, L]$ along the x-axis in the (x, y)-plane

$$\rho \ddot{\boldsymbol{r}} = \frac{\partial}{\partial x}((T_0 + E\varepsilon)\boldsymbol{\tau}), \quad \varepsilon = \left|\frac{\partial \boldsymbol{r}}{\partial x}\right| - 1, \quad \boldsymbol{\tau} = \frac{\partial \boldsymbol{r}}{\partial x} / \left|\frac{\partial \boldsymbol{r}}{\partial x}\right|$$

where $\boldsymbol{r}(x, t)$ is the position vector of the material point x at time t, ρ (linear density) and E (the Young's modulus) are constant. Write the linearized equations for $\boldsymbol{u}(x, t) = (u_1(x, t), u_2(x, t))$, assuming that $\boldsymbol{r}(x, t) = (x, 0) + \epsilon \boldsymbol{u}$ and $|\epsilon|$ is small. Show all work.

2. Consider the problem of minimizing the perimeter

$$\int_0^1 \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2} dt$$

of a simple closed curve with parametric equations

$$\begin{cases} x = x(t), \\ y = y(t) \end{cases}, \quad t \in [0, 1]$$

subject to the area constraint

$$2A = \int_0^1 (x(t)\dot{y}(t) - y(t)\dot{x}(t))dt.$$

Observe that the rotations $\boldsymbol{R}_\theta$ through any angle θ leave the Lagrangian

$$L(x, y, \dot{x}, \dot{y}) = \sqrt{\dot{x}^2 + \dot{y}^2} - \lambda(x\dot{y} - y\dot{x})$$

invariant. Use Noether's theorem to find the conserved quantity corresponding to this symmetry. Show all work.

3. Consider the initial value problem

$$\begin{cases} \ddot{x} - \epsilon(1 - x^2)\dot{x} + x = 0, \\ x(0) = 0, \\ \dot{x}(0) = 1. \end{cases}$$

when $|\epsilon|$ is small. Apply the two-scale expansion method to obtain uniform in time $O(\epsilon)$ approximation to the solution of the intial value problem above.