**Applied Mathematics Qualifying Written Exam**

**PART I. Do three of the following four problems.**

1. Compute the adjoint operator $L^*$, including boundary conditions, for the operator

$$L = \frac{d^2}{dx^2} + 4\frac{d}{dx} - 3, \quad y'(a) + 4y(a) = 0, \; y'(b) + 4y(b) = 0,$$

with $x \in [a \; b]$.

2. Given the equation

$$\frac{d^2y}{dt^2} + \epsilon\left(\frac{dy}{dt}\right)^3 + y = 0, \quad y(0) = 0, \quad \frac{dy}{dt}(0) = 1,$$

verify the asymptotic approximation

$$y = \left(1 + \frac{3\epsilon t}{4}\right)^{-1/2} \sin t + O(\epsilon).$$

3. Construct a phase portrait for the equation

$$\frac{dy}{dt} = \sin(y^2).$$

4. Compute the limit of solution $y(x,t)$ as $t \to \infty$ of the equation

$$\frac{\partial y}{\partial t} = \kappa\frac{\partial^2 y}{\partial x^2}, \qquad \frac{\partial y}{\partial x}(0,t) = \frac{\partial y}{\partial x}(1,t) = 0, \tag{1}$$

with $y(x,0) = x$, for $x \in [0 \; 1]$.

**PART II. Do two of the following three problems.**

1. Compute the Fourier sine and cosine series for the function

$$f(x) = x, \ \ x \in [0 \ 1].$$

   Compare convergence rates of the resulting series, and explain the difference.

2. Suppose that $f$ is given and that $u$ minimizes

$$\int\int_D \left(\frac{1}{2}|\nabla u|^2 + f(x,y)u\right) dx\, dy$$

   over smooth functions, where $D$ is a smooth, bounded domain. Assuming $u$ and $f$ are adequately smooth, show that

$$\nabla^2 u = f.$$

3. Nondimensionalize the "fluctuating" oscillator equation

$$\frac{d^2y}{dt^2} + \beta(\sin \omega t)^2 y = 0. \tag{2}$$

   with $\beta > 0$ and with initial conditions $y(0) = A$, $\dot{y}(0) = B$. Identify any non-dimensional numbers and interpret their significance. Define a notion of fast and of slow fluctuations, and solve the fast fluctuation case to 0th order.