PART I. Do three of the following four problems.

1. A spring oscillating in a viscous medium has the equation of motion

\[ m \frac{d^2 y}{dt^2} = -\mu \left( \frac{dy}{dt} \right)^2 - ky \]

with \( m, k, \mu > 0 \) and with initial conditions \( y(0) = A, \dot{y}(0) = 0 \). Suppose \( \mu \ll m \).

Scale to obtain a non-dimensional version and identify a small parameter \( \epsilon \).

2. Consider the equation

\[ \ddot{\theta} + \sin \theta = 0. \]

Determine the stability of the solution \( \theta(t) = 0 \) to linear perturbation.

3. Determine the Euler-Lagrange equation for a functional of the form

\[ J[y] = \int_0^1 F(x, y, y', y'') \, dx. \]

4. Find the Green’s function for the problem

\[ u'' - u = f(x), \quad 0 < x < 1, \]

with boundary conditions \( u(0) = u(1) = 0 \).
PART II. Do two of the following three problems.

1. Obtain the first two terms of asymptotic in $\epsilon$ approximations to all three solutions of the equation

$$\epsilon x^3 - 3x + 1 = 0.$$ 

2. Consider the system

$$\dot{x} = y + \mu x,$$
$$\dot{y} = x - x^2,$$

with parameter $\mu$. Find all bifurcations (in $\mu$) and sketch a bifurcation diagram (in $\mu$).

3. Consider the equation

$$u_{xx} + u_{yy} = 0$$

in the upper half plane $y > 0$, and

(a) the Dirichlet data $u(x, 0) = f(x),$
(b) the Neumann data $u_y(x, 0) = g(x),$

where $f$ and $g$ are $2\pi$ periodic in $x$. Assume $u$ is bounded at infinity and also is $2\pi$ periodic in $x$. Find the Fourier transform

$$L\hat{f}(k) = \hat{g}(k)$$

of the Dirichlet-to-Neumann map, where $\hat{f}, \hat{g}$ are the Fourier transforms of $f, g$, respectively.