1 Part I: do 3 of 4

1. Suppose that $\rho(x, t)$ and $v(x, t)$ solve the compressible Euler system

$$\begin{aligned}
\frac{\partial v}{\partial t} + (\nabla v)v &= -\frac{\nabla (p(\rho))}{\rho}, \\
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) &= 0,
\end{aligned}$$

where $p(\rho) = \rho^{7/5}$. Assume that the flow is irrotational i.e. $v(x, t) = \nabla \phi(x, t)$ for some scalar function $\phi$. For what function $Q(\rho)$ is the quantity

$$E(x, t) = Q(\rho(x, t)) + \frac{\partial \phi}{\partial t}(x, t) + \frac{1}{2} |\nabla \phi(x, t)|^2$$

independent of $x$? Prove your assertion.

2. The period $T$ of revolution of a planet moving along a circular orbit around a star depends only on the planet’s distance to the star $R$ and the star’s mass $M$. Use dimensional analysis to find Kepler’s formula for the ratio of the periods $T_1/T_2$ for any two such planets revolving around the same star.

3. Use scaling to express the wave function $\psi(x, t)$, solving Schrödinger’s equation

$$i\hbar \psi_t = -\frac{\hbar^2}{2m} \Delta \psi - \frac{e^2}{4\pi \varepsilon_0 |x|} \psi, \quad x \in \mathbb{R}^3, \ t > 0$$

in terms of the solution $\phi(\xi, \tau)$ of

$$i\phi_\tau = -\Delta \phi - \frac{\phi}{|\xi|}, \quad \xi \in \mathbb{R}^3, \ \tau > 0.$$ 

Give expressions for the characteristic time and length in terms of the positive constants $\hbar, m, e$ and $\varepsilon_0$.

4. Assume that the electric and magnetic fields $E(x, t), B(x, t), x \in \mathbb{R}^3, t > 0$ are given by

$$E(x, t) = \begin{cases} E^+, & x \cdot n > 0, \ t > 0 \\
E^-, & x \cdot n < 0, \ t > 0 \end{cases}, \quad B(x, t) = \begin{cases} B^+, & x \cdot n > 0, \ t > 0 \\
B^-, & x \cdot n < 0, \ t > 0 \end{cases}$$

where $n$ is the unit normal to the plane $x \cdot n = 0$ separating two media. Suppose that the vectors $E^\pm, B^\pm$ are constant. What extra conditions do these four vectors have to satisfy in order for the following two Maxwell’s equations

$$\begin{aligned}
\nabla \times E &= -\frac{1}{c} \frac{\partial B}{\partial t}, \\
\nabla \cdot B &= 0
\end{aligned}$$

to hold in $\mathbb{R}^3 \times (0, +\infty)$ in the sense of distributions?
2  Part II: do 2 of 3

1. Suppose that a point mass $m$ in a plane is attached by two identical springs, each with spring constant $k$, to two given points $A$ and $B$ in the plane. Assume that all springs are at their natural length when the point mass is at the midpoint of $AB$. Also assume that there is no gravity and no friction.

(a) Identify degrees of freedom.

(b) Write the system’s Lagrangian.

(c) Write the Euler-Lagrange equations of motion.

2. Compute the Green’s function for the boundary value problem

$$\begin{cases}
x^2 u''(x) + xu'(x) + u(x) = f(x), & x \in [1, e^\pi] \\
u(1) = 0, \\
u'(e^\pi) = u(e^\pi).
\end{cases}$$

Write the solution $u(x)$ in terms of this Green’s function.

3. Use Laplace’s method to find the first two terms of the asymptotic expansion of

$$I(x) = \int_0^1 t^x \sin^2 t dt,$$

as $x \to +\infty$. 