Applied Mathematics Qualifying Written Exam (August 2016)

1 Part I: do 3 of 4

1. Suppose that $\rho(\boldsymbol{x},t)$ and $\boldsymbol{v}(\boldsymbol{x},t)$ solve the compressible Euler system

$$\begin{cases} \dfrac{\partial \boldsymbol{v}}{\partial t} + (\nabla \boldsymbol{v})\boldsymbol{v} = -\dfrac{\nabla(p(\rho))}{\rho}, \\ \dfrac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0, \end{cases}$$

where $p(\rho) = \rho^{7/5}$. Assume that the flow is irrotational i.e. $\boldsymbol{v}(\boldsymbol{x},t) = \nabla\phi(\boldsymbol{x},t)$ for some scalar function ϕ. For what function $Q(\rho)$ is the quantity

$$E(\boldsymbol{x},t) = Q(\rho(\boldsymbol{x},t)) + \frac{\partial \phi}{\partial t}(\boldsymbol{x},t) + \frac{1}{2}|\nabla\phi(\boldsymbol{x},t)|^2$$

independent of $\boldsymbol{x}$? Prove your assertion.

2. The period T of revolution of a planet moving along a circular orbit around a star depends only on the planet's distance to the star R and the star's mass M. Use dimensional analysis to find Kepler's formula for the ratio of the periods T_1/T_2 for any two such planets revolving around the same star.

3. Use scaling to express the wave function $\psi(\boldsymbol{x},t)$, solving Schrödinger's equation

$$i\hbar\psi_t = -\frac{\hbar^2}{2m}\Delta\psi - \frac{e^2}{4\pi\varepsilon_0|\boldsymbol{x}|}\psi, \qquad \boldsymbol{x} \in \mathbb{R}^3,\ t > 0$$

in terms of the solution $\phi(\boldsymbol{\xi},\tau)$ of

$$i\phi_\tau = -\Delta\phi - \frac{\phi}{|\boldsymbol{\xi}|}, \qquad \boldsymbol{\xi} \in \mathbb{R}^3,\ \tau > 0.$$

Give expressions for the characteristic time and length in terms of the positive constants $\hbar$, m, e and ε_0.

4. Assume that the electric and magnetic fields $\boldsymbol{E}(\boldsymbol{x},t)$, $\boldsymbol{B}(\boldsymbol{x},t)$, $\boldsymbol{x} \in \mathbb{R}^3$, $t > 0$ are given by

$$\boldsymbol{E}(\boldsymbol{x},t) = \begin{cases} \boldsymbol{E}^+, & \boldsymbol{x}\cdot\boldsymbol{n} > 0,\ t > 0 \\ \boldsymbol{E}^-, & \boldsymbol{x}\cdot\boldsymbol{n} < 0,\ t > 0 \end{cases}, \qquad \boldsymbol{B}(\boldsymbol{x},t) = \begin{cases} \boldsymbol{B}^+, & \boldsymbol{x}\cdot\boldsymbol{n} > 0,\ t > 0 \\ \boldsymbol{B}^-, & \boldsymbol{x}\cdot\boldsymbol{n} < 0,\ t > 0 \end{cases},$$

where $\boldsymbol{n}$ is the unit normal to the plane $\boldsymbol{x}\cdot\boldsymbol{n} = 0$ separating two media. Suppose that the vectors $\boldsymbol{E}^\pm$, $\boldsymbol{B}^\pm$ are constant. What extra conditions do these four vectors have to satisfy in order for the following two Maxwell's equations

$$\begin{cases} \nabla \times \boldsymbol{E} = -\dfrac{1}{c}\dfrac{\partial \boldsymbol{B}}{\partial t}, \\ \nabla \cdot \boldsymbol{B} = 0 \end{cases}$$

to hold in $\mathbb{R}^3 \times (0,+\infty)$ in the sense of distributions?

2 Part II: do 2 of 3

1. Suppose that a point mass m in a plane is attached by two identical springs, each with spring constant k, to two given points A and B in the plane. Assume that all springs are at their natural length when the point mass is at the midpoint of AB. Also assume that there is no gravity and no friction.
   (a) Identify degrees of freedom.
   (b) Write the system's Lagrangian.
   (c) Write the Euler-Lagrange equations of motion.

2. Compute the Green's function for the boundary value problem

$$\begin{cases} x^2u''(x) + xu'(x) + u(x) = f(x), \;\; x \in [1, e^\pi] \\ u(1) = 0, \\ u'(e^\pi) = u(e^\pi). \end{cases}$$

   Write the solution $u(x)$ in terms of this Green's function.

3. Use Laplace's method to find the first two terms of the asymptotic expansion of

$$I(x) = \int_0^1 t^x \sin^2 t dt,$$

   as $x \to +\infty$.