Applied Mathematics Exam
August 12, 2014

PART I. Do three of the following four problems.

1. Find inner, outer, and uniform (or composite) approximations to the solution of

$$\epsilon y'' + (1+x)y' + y = 0, \quad y(0) = 0, \quad y(1) = 1.$$

   Assume that ϵ is small and that there is a boundary layer at $x = 0$.

2. Find a scaling of x, t, and u so that the equation

$$u_t = au_x - bu^3 + Du_{xx},$$

   where a, b, and D are non-zero constants, has no free parameters.

3. Draw the phase plane for the equation

$$\ddot{\theta} + \sin\theta = 0.$$

   Find all equilibria and characterize their stability.

4. Find the extremal curve for the functional

$$J[y] = \int_0^1 ((y')^2 + 3y + 2x)dx$$

   subject to the boundary conditions $y(0) = 0$, $y(1) = 1$.

PART II. Do two of the following three problems.

1. Obtain a periodic approximation with $O(\epsilon^2)$ error to the solution of

$$\ddot{y} + 9y - 3\epsilon y^3 = 0, \quad y(0) = 0, \quad \dot{y}(0) = 1.$$

2. Use separation of variables to solve the problem

$$u_t + u_x = u_{xx} \quad 0 < t,\ 0 < x < 1,$$

with $u(0,t) = u(1,t) = 0$ for $t > 0$ and $u(x,0) = f(x)$ where f is a given, smooth function. How does the solution behave as $t \to \infty$?

3. Using Fourier transforms, find the solution to the problem

$$iu_t = u_{xx}, \quad u(x,0) = f(x),$$

with $-\infty < x < \infty$ and f smooth and integrable on $-\infty < x < \infty$ (here $i = \sqrt{-1}$). Show that

$$|u(x,t)| \le \frac{C}{\sqrt{t}} \int_{-\infty}^{\infty} |u(\hat{x},0)| d\hat{x}$$

for some constant C.