Comprehensive Examination in Algebra
Department of Mathematics, Temple University

January 2021

Part I. Do three of these problems.

I.1 An abelian group A, written additively, is called *divisible* if $A = \{na : a \in A\}$ for all $0 \neq n \in \mathbb{Z}$ and *torsion* if every $a \in A$ has a finite order. Now let $A = (\mathbb{Q}, +)$. Prove:

a) If B is any nonzero subgroup of A, then A/B is both divisible and torsion.

b) A has no proper subgroups of finite index.

I.2 Let V be a finite-dimensional vector space over some field K, let $T \in \operatorname{End}_K(V)$, and let $W \subseteq V$ be a subspace such that $T(W) \subseteq W$. Let m, m_1 and m_2 denote the minimal polynomials of T viewed as an operator on V, W, and V/W, respectively. Show:

a) m divides $m_1 m_2$.

b) If m_1 and m_2 are relatively prime, then $m = m_1 m_2$.

c) Give an example with $m \neq m_1 m_2$.

I.3 Consider the four rings $R_n := \mathbb{Q}[x]/(x^2 - n)$ with $n \in \{1, 2, 3, 4\}$.

a) Which of these rings are isomorphic?

b) Which are fields?

Please justify your answers.

I.4 Let $E \supseteq F$ be a field extension and let $\alpha, \beta \in E$ be algebraic over F. Prove that α and β have the same minimal polynomial over F if and only if there exists an F-isomorphism $\varphi : F(\alpha) \xrightarrow{\sim} F(\beta)$ such that $\varphi(\alpha) = \beta$.

Part II. Do two of these problems.

II.1 Let G be a group of order $120 = 2^3 \cdot 3 \cdot 5$. Show that G either has a normal Sylow 5-subgroup or a (normal) subgroup of index 2.

II.2 Let R be a commutative ring, with $1 \neq 0$ but not necessarily an integral domain, and let $A \in \mathrm{Mat}_{n\times k}(R)$ with $n < k$. Prove that the columns of A are linearly dependent over R, that is, there exists a non-zero column $v \in R^k$ such that Av is the zero column in R^n.

II.3 Let K be an extension field of $\mathbb{Q}$ with $[K : \mathbb{Q}] = n < \infty$. Show:

a) There are n distinct embeddings $\sigma_i \colon K \hookrightarrow \mathbb{C}$.

b) Let $\alpha \in K$ be given. Then the distinct members of $\{\sigma_1(\alpha), \ldots, \sigma_n(\alpha)\}$ are the eigenvalues of the linear operator $A \in \mathrm{End}_{\mathbb{Q}}(K)$ that is defined by $A(\beta) = \alpha\beta$.